

Introduction:

Transform techniques are an important tool in the analysis of signals and systems.

Fourier series: (CTFS and DTFS)
→ For the analysis of periodic signals.

Fourier transform: (DTFS and CTFT).
→ For the analysis of aperiodic signals.

Simple and systematic transforms:

Laplace transform:
→ For the analysis of continuous time signals and systems.

Z-transform:
→ For the analysis of discrete time signals and systems.

* The Laplace transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself. To solve the differential eqns which are in time domain, they are first converted into algebraic eqns in frequency domain using Laplace transform, the algebraic eqns are manipulated in s-domain and the result obtained in s-domain is converted back into time domain using inverse Laplace transform.

Z-transform:

Z-transform has the advantage that it is a simple and systematic method and the complete soln. can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself. To solve difference eqns which are in time domain, they are converted first into algebraic eqns. in z-d using z-transform, the algebraic eqns. are manipulated in z-domain and the result obtained is converted back into time domain using inverse z-transform.

The bilateral or two sided z-transform of a discrete time signal $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

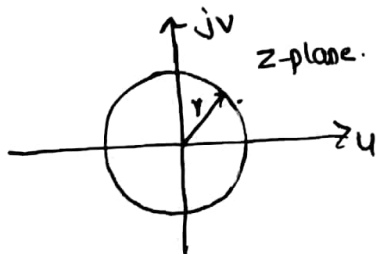
The unilateral or one sided z-transform of $x(n)$ is defined

where z is a complex variable, as $x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$Z = u + jv = r e^{j\Omega}$$

$\downarrow$ real part of z $\downarrow$ imag. part of z

magnitude of z , $r = \sqrt{u^2 + v^2}$
phase of z , $\Omega = \tan^{-1}(v/u)$



A 2-D complex plane with values of u on horizontal axis and the values of v on vertical axis as shown in figure is called Z-plane.

* The set of z values for which the summation converges is called region of convergence (ROC) for the transform.

Region of convergence:

Since z -transform is an infinite power series, it exists only for those values of z for which the series converges. The ROC of $x(z)$ is the set of all values of z for which $x(z)$ attains a finite value.

z -transform and ROC of finite duration sequence

Right sided sequence:

Consider the sequence $x(n) = \overset{x(0)}{1}, \overset{x(1)}{2}, \overset{x(2)}{2}, \overset{x(3)}{1}$

$$\begin{aligned} x(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^3 x(n) z^{-n} \\ &= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} \\ &= 1 + 2z^{-1} + 2z^{-2} + z^{-3} \end{aligned}$$

In the above summation when $z=0$, all the terms except the first term become infinite. i.e. $x(z)$ converges for all values of z except at $z=0$.

$\therefore$ The ROC for finite duration right sided signal is entire z -plane except at $z=0$.
Mathematically ROC: $|z| > 0$

Left sided sequence:

Consider the sequence $x(n) = \overset{x(-3)}{1}, \overset{x(-2)}{2}, \overset{x(-1)}{1}, \overset{x(0)}{3}$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-3}^0 x(n) z^{-n}$$

$$X(z) = x(-3)z^3 + x(-2)z^2 + x(-1)z^1 + x(0)$$

$$= z^3 + 2z^2 + z + 3$$

In the above summation when $z = \infty$, all the terms except the last term become infinite. i.e. $x(z)$ converges for all values of z except at $z = \infty$.
 $\therefore$ The ROC for finite duration left sided

Signal is entire z -plane except at $z = \infty$
 mathematically ROC: $|z| < \infty$

Double Sided Sequence: (Two sided sequence)

A signal that has finite duration on both left and right sides is known as double sided. In this case the ROC is the entire z -plane except at $z = 0$ and $z = \infty$.

Consider the sequence $x(n) = (2, 1, 1, 2)$
 $\uparrow$
 $x(-2) \quad x(-1) \quad x(0) \quad x(1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-2}^1 x(n)z^{-n}$$

$$= x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1}$$

$$= 2z^2 + z + 1 + 2z^{-1}$$

The above expression for $x(z)$ becomes infinity at $z = 0$ and $z = \infty$. Hence the ROC is the

entire z -plane except at $z = 0$ and $z = \infty$

This is explained mathematically by writing the ROC as, ROC: $0 < |z| < \infty$.

Z-transform and ROC of infinite duration sequence:

Right sided (positive time exponential) sequence:

A right sided infinite sequence is defined as

$$x(n) = a^n \quad n \geq 0$$

$$\text{i.e. } x(n) = a^n u(n)$$

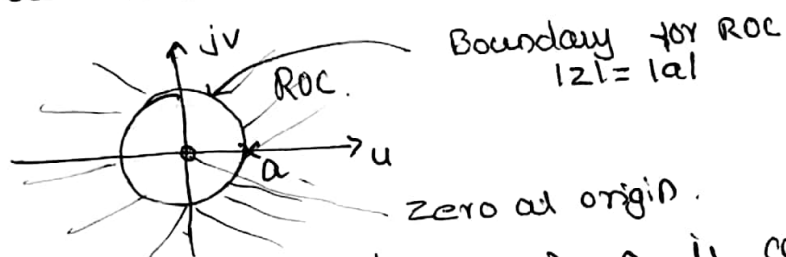
$$\begin{aligned}
 X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
 &= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}
 \end{aligned}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \underline{\underline{\frac{z}{z-a}}}$$

$X(z)$ converges if $|az^{-1}| < 1$

$$\frac{|a|}{|z|} < 1 \Rightarrow |z| > |a|$$

i.e. $X(z)$ converges for all points external to the circle of radius ' a ' in z -plane. $\therefore$ the ROC of $X(z)$ is exterior of the circle of radius ' a ' in z -plane as shown in fig. given below.



* Values of z for which $X(z) = 0$ is called zeros of $X(z)$ while the values of z for which $X(z) = \infty$ is called poles of $X(z)$. poles are usually indicated by 'x' at $z = a$ and zeros by 'o' at $z = 0$

Left sided (negative time exponential) sequence:

A left sided infinite sequence is defined as

$$x(n) = -b^n u(-n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

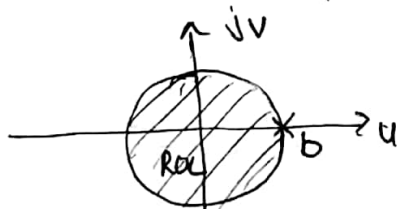
$$= \sum_{n=-\infty}^{\infty} -b^n u(-n) z^{-n} = - \sum_{n=-\infty}^0 b^n z^{-n} = - \sum_{n=\infty}^0 b^n z^n$$

$$= - \sum_{n=0}^{\infty} (b^{-1} z)^n = - \frac{1}{1 - b^{-1} z} = \underline{\underline{\frac{b}{z-b}}}$$

$x(z)$ converges $|b'z| < 1$

$$\frac{|z|}{|b|} < 1 \Rightarrow |z| < |b|$$

i.e. $x(z)$ converges for all points internal to the circle of radius ' b '. $\therefore$ The ROC of $x(z)$ is the interior of the circle of radius ' b ' in the z -plane.



Double sided sequence:

$$x(n) = a^n u(n) - b^n u(-n) \Rightarrow x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} - \sum_{n=-\infty}^{\infty} b^n u(-n) z^{-n}$$

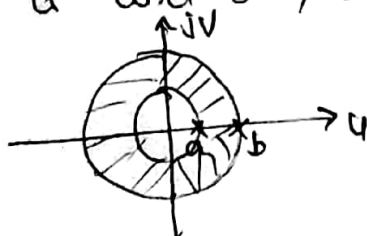
$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=-\infty}^0 b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=0}^{\infty} (b^{-1}z)^n$$

$$= \frac{1}{1-az^{-1}} - \frac{1}{1-b^{-1}z}$$

The 1st term converges if $|az^{-1}| < 1 \Rightarrow |z| > |a|$
 The 2nd term converges if $|b^{-1}z| < 1 \Rightarrow |z| < |b|$

$\therefore$ ROC is the region between two circles of radius ' a ' and ' b ', where $|b| > |a|$



Summary:

- Sequence
- 1) finite right sided
 - 2) finite left sided
 - 3) finite double sided
 - 4) infinite right sided
 - 5) infinite left sided
 - 6) infinite double sided.

- ROC.
- entire z -plane except at $z=0$
- entire z -plane except at $z=\infty$
- entire z -plane ~~except~~ except at $z=0$ & $z=\infty$
- exterior of the circle of radius a' , $|z| > |a|$.
- interior of the circle of radius b' , $|z| < |b|$
- Region b/w the two circles of radius a' & b' , where $|b| > |a|$, and $|a| < |z| < |b|$

✓ Properties of ROC:

- 1) The ROC is a ring in the z -plane centered at the origin.
- 2) The ROC cannot contain any poles.
- 3) If $x(n)$ is a finite right sided sequence, then the ROC is the entire z -plane except at $z=0$.
- 4) If $x(n)$ is a finite left sided sequence, then the ROC is the entire z -plane except at $z=\infty$.
- 5) If $x(n)$ is a finite double sided sequence, then the ROC is the entire z -plane except at $z=0$ and $z=\infty$.
- 6) If $x(n)$ is an infinite double sided sequence, then the ROC will consist of a ring in the z -plane bounded on the interior and exterior by a pole.
- 7) The ROC of a stable system contains the unit circle.
- 8) The ROC must be a connected region.

Z-transform of

Pod $z = e^{j\Omega}$

Problems:

a) $\text{sec}(n) = u(n)$

b) $x(n) = \delta(n)$

c) $x(n) = e^{j\omega_0 n} u(n)$

d) $x(n) = u(n)$

e) $x(n) = a^{|n|}$ for $|a| < 1$ and $|a| > 1$

f) $x(n) = a^{-n} u(n-1)$

g) $x(n) = u(n) - u(n-6)$

k) $x(n) = (2, -1, 0, 3, 4)$

i) $\text{span}_2(1, -2, 3, -1, 2)$

j) $x(n) = (5, 3, -2, 0, 4, -3)$

Ans:

a) $x(n) = u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

$$X(z) \text{ converges if } |z^{-1}| < 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |z| > 1$$

$$\text{Roc: } |z| > 1.$$

b) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = z^{-n} \Big|_{n=0}$$

$$= 1 \cdot z^0 \quad \text{Roc: entire } z \text{ plane.}$$

c) $x(n) = e^{j\Omega_0 n} u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} = \sum_{n=0}^{\infty} (e^{j\Omega_0} z^{-1})^n = \frac{1}{1 - e^{j\Omega_0} z^{-1}}$$

$$= \frac{z}{z - e^{j\Omega_0}}$$

$$X(z) \text{ converges if } |e^{j\Omega_0} z^{-1}| < 1 \Rightarrow \frac{|e^{j\Omega_0}|}{|z|} < 1$$

$$|z| > |e^{j\Omega_0}| \Rightarrow |z| > 1$$

$$\text{Roc: } |z| > 1$$

$$\begin{aligned} e^{j\Omega_0} &= \cos \Omega_0 + j \sin \Omega_0 \\ |e^{j\Omega_0}| &= \sqrt{\cos^2 \Omega_0 + \sin^2 \Omega_0} \\ &= \sqrt{1} = 1 \end{aligned}$$

d) $x(n) = u(-n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} u(-n) z^{-n} = \sum_{n=-\infty}^{\infty} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}$$

$$X(z) \text{ converges if } |z| < 1$$

$$\therefore \text{Roc: } |z| < 1$$

$$e) x(n) = a^{|n|}$$

$$|n| = \begin{cases} n & \text{for } n \geq 0 \\ -n & \text{for } n < 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (a^{-n} z^{-n}) + \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \sum_{n=-\infty}^{-1} a^n z^n + \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \sum_{n=1}^{\infty} (a z)^n + \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \left(\sum_{n=0}^{\infty} (a z)^n - 1 \right) + \sum_{n=0}^{\infty} (a z^{-1})^n$$

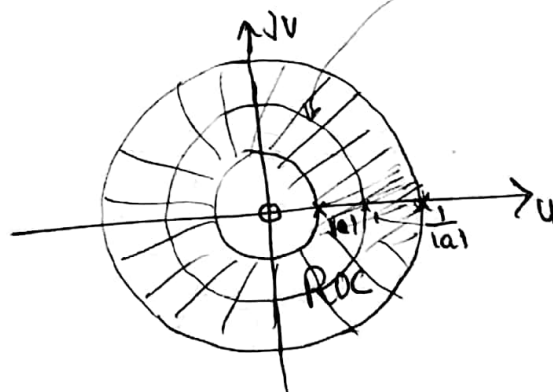
$$= \left(\frac{1}{1-a z} - 1 \right) + \frac{1}{1-a z^{-1}}$$

$$= \frac{1-(1-a z)}{1-a z} + \frac{z}{z-a}$$

$$= \frac{a z}{1-a z} + \frac{z}{z-a}$$

$$= \frac{a z(z-a) + z(1-a z)}{(1-a z)(z-a)}$$

$$Roc: |a| < |z| < \frac{1}{|a|}$$



Zeros: $z=0$

Poles: $az=1$ and $z=a$
 $z = \frac{1}{a}$

$$f) x(n) = a^{-n} u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^{-n} u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} = \sum_{n=\infty}^1 a^n z^n$$

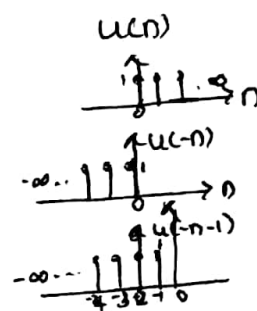
$$= \sum_{n=1}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n - 1$$

$$= \frac{1}{1-az} - 1 = \frac{1-(1-az)}{1-az}$$

$$= \frac{az}{1-az} = \frac{-z}{z-\frac{1}{a}}$$

$$X(z) \text{ converges } |az| < 1$$

$$\text{ROC: } |z| < \frac{1}{|a|}$$



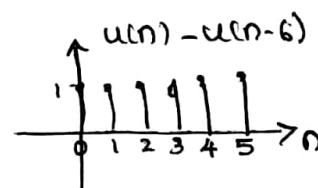
$$g) x(n) = u(n) - u(n-6)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [u(n) - u(n-6)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) z^{-n} - \sum_{n=-\infty}^{\infty} u(n-6) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} - \sum_{n=6}^{\infty} z^{-n}$$

$$= \sum_{n=0}^5 z^{-n} = \underbrace{1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}}_{\text{finite right sided seq.}}$$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^5 x(n) z^{-n} = \sum_{n=0}^5 z^{-n}$$

$$\therefore \text{ROC: Entire } z\text{-plane except at } z=0$$

Qn. Find z-transform of $x(n) = \cos \Omega n u(n)$

$$= \left[\frac{e^{j\Omega n}}{2} + \frac{e^{-j\Omega n}}{2} \right] u(n)$$

W.K.T $z \{ e^{j\Omega n} u(n) \} = \frac{1}{1 - e^{j\Omega} z^{-1}}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left[\frac{e^{j\Omega n}}{2} + \frac{e^{-j\Omega n}}{2} \right] u(n) z^{-n}$$

$$= \frac{1}{2} \underbrace{\sum_{n=-\infty}^{\infty} e^{j\Omega n} u(n) z^{-n}}_{\frac{1}{1 - e^{j\Omega} z^{-1}}} + \frac{1}{2} \underbrace{\sum_{n=-\infty}^{\infty} e^{-j\Omega n} u(n) z^{-n}}_{\frac{1}{1 - e^{-j\Omega} z^{-1}}}$$

$$\therefore X(z) = \frac{1}{2} \left[\frac{1}{1 - e^{j\Omega} z^{-1}} + \frac{1}{1 - e^{-j\Omega} z^{-1}} \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - e^{j\Omega}} + \frac{z}{z - e^{-j\Omega}} \right]$$

$$= \frac{1}{2} \left[\frac{z(z - e^{-j\Omega}) + z(z - e^{j\Omega})}{(z - e^{j\Omega})(z - e^{-j\Omega})} \right]$$

$$= \frac{1}{2} \left[\frac{z^2 - z e^{-j\Omega} + z^2 - z e^{j\Omega}}{z^2 - z e^{-j\Omega} - z e^{j\Omega} + 1} \right]$$

$$= \frac{1}{2} \left[\frac{2z^2 - z(e^{j\Omega} + e^{-j\Omega})}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2} \left[\frac{z(2z - (e^{j\Omega} + e^{-j\Omega}))}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2} \left[\frac{z(2z - 2\cos \Omega)}{z^2 - 2z\cos \Omega + 1} \right] = \frac{z(z - \cos \Omega)}{z^2 - 2z\cos \Omega + 1}$$

ROC: $|z| > 1$

Qn. Find the z-transform of $x(n) = \sin n\Omega u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \frac{1}{2j} \sum_{n=-\infty}^{\infty} [e^{jn\Omega} - e^{-jn\Omega}] u(n) z^{-n}$$

$$= \frac{1}{2j} \left\{ \underbrace{\sum_{n=-\infty}^{\infty} e^{jn\Omega} u(n) z^{-n}}_{\frac{z}{z - e^{j\Omega}}} + \underbrace{\sum_{n=-\infty}^{\infty} e^{-jn\Omega} u(n) z^{-n}}_{\frac{z}{z - e^{-j\Omega}}} \right\}$$

$$= \frac{1}{2j} \left[\frac{z}{z - e^{j\Omega}} - \frac{z}{z - e^{-j\Omega}} \right] = \frac{1}{2j} \left[\frac{z(z - e^{-j\Omega}) - z(z - e^{j\Omega})}{(z - e^{j\Omega})(z - e^{-j\Omega})} \right]$$

$$= \frac{1}{2j} \left[\frac{z(z - e^{-j\Omega} - z + e^{j\Omega})}{z^2 - ze^{-j\Omega} - ze^{j\Omega} + 1} \right]$$

$$= \frac{1}{2j} \left[\frac{z(e^{j\Omega} - e^{-j\Omega})}{z^2 - z(e^{j\Omega} + e^{-j\Omega}) + 1} \right]$$

$$= \frac{1}{2j} \left[\frac{z \cdot 2j \sin \Omega}{z^2 - 2z \cos \Omega + 1} \right] = \frac{z \sin \Omega}{z^2 - 2z \cos \Omega + 1}$$

ROC: $|z| > 1$

Qn. Find the Z transform and ROC of $x(z)$ for $x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(-\frac{1}{3}\right)^n u(n)$. Also find the pole-zero location.

w.k.T $a^n u(n) \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}$

$\therefore \left(\frac{5}{7}\right)^n u(n) \xleftrightarrow{Z} \frac{1}{1 - \frac{5}{7}z^{-1}}$

$\left(-\frac{1}{3}\right)^n u(n) \xleftrightarrow{Z} \frac{1}{1 - (-\frac{1}{3})z^{-1}} = \frac{1}{1 + \frac{1}{3}z^{-1}}$

$\therefore x(z) = 3 \cdot \underbrace{\frac{1}{1 - \frac{5}{7}z^{-1}}}_{x_1(z)} + 2 \cdot \underbrace{\frac{1}{1 + \frac{1}{3}z^{-1}}}_{x_2(z)}$

$1 - \frac{10}{7}z^{-1} = \frac{z - \frac{10}{7}}{z}$

$= 3 \frac{z}{z - \frac{5}{7}} + 2 \frac{z}{z + \frac{1}{3}}$

$= \frac{3z(z + \frac{1}{3}) + 2z(z - \frac{5}{7})}{(z - \frac{5}{7})(z + \frac{1}{3})} = \frac{3z^2 + z + 2z^2 - \frac{10}{7}z}{(z - \frac{5}{7})(z + \frac{1}{3})}$

$\frac{0.71}{1.58} = \frac{10}{5}$

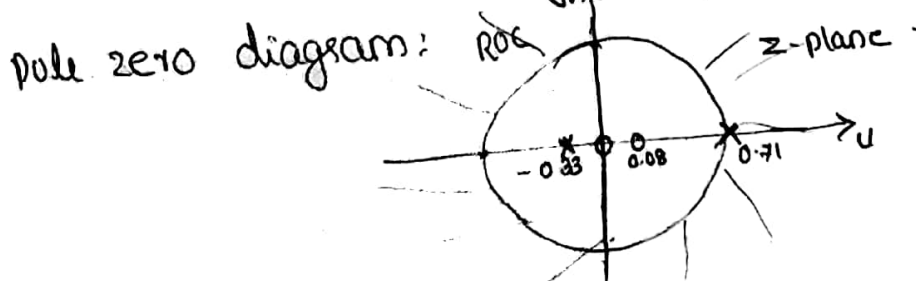
$= \frac{5z^2 - \frac{3}{7}z}{(z - \frac{5}{7})(z + \frac{1}{3})} = \frac{z(5z - \frac{3}{7})}{(z - \frac{5}{7})(z + \frac{1}{3})}$

$\frac{35}{1300} = \frac{35}{1300}$

$x_1(z)$ converges if $|\frac{5}{7}z^{-1}| < 1 \Rightarrow |z| > \frac{5}{7}, |z| > 0.71$
 $x_2(z)$ converges if $|\frac{1}{3}z^{-1}| < 1 \Rightarrow |z| > \frac{1}{3}, |z| > 0.33$
 $\therefore x(z)$ converges if $|z| > 0.71 \therefore \text{ROC: } |z| > 0.71$

poles: $z = \frac{5}{7} = \underline{0.71}$ and $z = -\frac{1}{3} = \underline{-0.33}$.

zeros: $z = 0$ and $5z = \frac{3}{7} \Rightarrow z = \frac{3}{35} = 0.08$



Properties of z-transform:

- 1) Linearity: If $x_1(n) \xleftrightarrow{Z} X_1(z)$
 $x_2(n) \xleftrightarrow{Z} X_2(z)$

Then ~~any~~ $x(n) = ax_1(n) + bx_2(n) \xleftrightarrow{Z} X(z) = aX_1(z) + bX_2(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)] z^{-n}$
 $= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$
 $= aX_1(z) + bX_2(z)$

- 2) Time shifting: If $x_1(n) \xleftrightarrow{Z} X_1(z)$

Then $x(n) = x_1(n-k) \xleftrightarrow{Z} X(z) = z^{-k} X_1(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n-k) z^{-n}$

Put $m = n-k \Rightarrow n = m+k$
 $\therefore X(z) = \sum_{m=-\infty}^{\infty} x_1(m) z^{-(m+k)} = \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} z^{-k}$
 $= z^{-k} \sum_{m=-\infty}^{\infty} x_1(m) z^{-m} = z^{-k} X_1(z)$

- 3) Time Reversal: If $x_1(n) \xleftrightarrow{Z} X_1(z)$

Then $x(n) = x_1(-n) \xleftrightarrow{Z} X(z) = X_1(z^{-1})$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(-n) z^{-n}$

Put $m = -n \Rightarrow n = -m$
 $\therefore X(z) = \sum_{m=-\infty}^{\infty} x_1(m) z^{-(-m)} = \sum_{m=-\infty}^{\infty} x_1(m) (z^{-1})^m$
 $= X_1(z^{-1})$

4. Multiplication by an exponential seq.

If $x_1(n) \xleftrightarrow{Z} X_1(z)$ Then $x(n) = a^n x_1(n) \xleftrightarrow{Z} X(z) = X_1(a^{-1}z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^n x_1(n) z^{-n}$
 $= \sum_{n=-\infty}^{\infty} x_1(n) (a^{-1}z)^{-n} = X_1(a^{-1}z)$

5. Multiplication by n (ramp) If $x_1(n) \xleftrightarrow{Z} X_1(z)$

Then $x(n) = nx_1(n) \xleftrightarrow{Z} X(z) = -z \frac{d}{dz} X_1(z)$

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} nx_1(n) z^{-n}$
 $= \sum_{n=-\infty}^{\infty} x_1(n) n \cdot z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n) (-n) z^{-n-1} = -z \sum_{n=-\infty}^{\infty} x_1(n) \frac{d}{dz} z^{-n}$
 $= -z \frac{d}{dz} \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = -z \frac{d}{dz} X_1(z)$

$$(OR) \quad X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \quad \text{or} \quad \sum_{n=-\infty}^{\infty} n x_1(n) z^{-n}$$

$$\frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} [x_1(n)] \frac{d}{dz} z^{-n} \Rightarrow \frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} x_1(n) (-n) z^{-n-1}$$

$$= \frac{dX_1(z)}{dz} = - \sum_{n=-\infty}^{\infty} n x_1(n) z^{-1} z^{-n} = -z^{-1} \sum_{n=-\infty}^{\infty} [n x_1(n)] z^{-n}$$

$$-z \frac{dX_1(z)}{dz} = \sum_{n=-\infty}^{\infty} [n x_1(n)] z^{-n}$$

$$\therefore n x_1(n) \xleftrightarrow{Z} -z \frac{dX_1(z)}{dz}$$

6. Convolution: If $x_1(n) \xleftrightarrow{Z} X_1(z)$ and $x_2(n) \xleftrightarrow{Z} X_2(z)$

$$\text{Then } x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

$$\text{Proof: } X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n}, \text{ changing the order of summation}$$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$$

$$\text{Put } m = n - k \Rightarrow n = m + k$$

$$\therefore X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m) z^{-(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m) z^{-m} z^{-k} = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{m=-\infty}^{\infty} x_2(m) z^{-m}$$

$$= X_1(z) X_2(z)$$

7. Initial value theorem: If $x(n) \xleftrightarrow{Z} X(z)$

$$\text{Then } x(0) = \lim_{z \rightarrow \infty} z X(z).$$

$$\text{Proof: } X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

As $z \rightarrow \infty$, all the terms ^{vanish} except $x(0)$.

$$\therefore \lim_{z \rightarrow \infty} z X(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x(n) z^{-n+1} = x(0) \text{ i.e. } x(0) = \lim_{z \rightarrow \infty} z X(z)$$

8. Final value theorem: If $x(n) \xleftrightarrow{Z} X(z)$, then

$$x(\infty) = \lim_{z \rightarrow 1} (z-1) X(z).$$

$$\text{Proof: } X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Form the eq. $x(n+1) - x(n)$ and take its Z-transform

$$\begin{aligned}
 Z\{x(n+1) - x(n)\} &= \sum_{n=0}^{\infty} [x(n+1) - x(n)] Z^{-n} \\
 &= [x(1) - x(0)] Z^{-0} + [x(2) - x(1)] Z^{-1} + [x(3) - x(2)] Z^{-2} \\
 &\quad + \dots + [x(\infty) - x(\infty-1)] Z^{-\infty+1}
 \end{aligned}
 \rightarrow \textcircled{1}$$

By using shifting property.

$$x(n+m) \xleftrightarrow{Z} Z^m [x(z) - \sum_{k=0}^{m-1} x(k) Z^{-k}]$$

$$x(n+1) \xleftrightarrow{Z} Z [x(z) - x(0)] = Zx(z) - x(0)$$

$$\begin{aligned}
 Z\{x(n+1) - x(n)\} &= Zx(z) - Zx(0) - x(z) \\
 &= (Z-1)x(z) - Zx(0) \rightarrow \textcircled{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore (Z-1)x(z) - Zx(0) &= [x(1) - x(0)] + [x(2) - x(1)] Z^{-1} \\
 &\quad + \dots + [x(\infty) - x(\infty-1)] Z^{-\infty}
 \end{aligned}$$

Taking limit $z \rightarrow 1$ on both sides.

$$\lim_{z \rightarrow 1} (Z-1)x(z) - Zx(0) = \lim_{z \rightarrow 1} \left\{ [x(1) - x(0)] + [x(2) - x(1)] Z^{-1} + \dots + [x(\infty) - x(\infty-1)] Z^{-\infty} \right\}$$

$$\begin{aligned}
 \lim_{z \rightarrow 1} (Z-1)x(z) - x(0) &= x(1) - x(0) + x(2) - x(1) + x(3) - x(2) \\
 &\quad + \dots + x(\infty) - x(\infty-1) \\
 &= x(\infty) - x(0)
 \end{aligned}$$

$$x(\infty) = \lim_{z \rightarrow 1} (Z-1)x(z) - x(0)$$

$$\begin{aligned}
 &\text{~~~~~} \\
 &= 0
 \end{aligned}$$

Time shifting (Time delay) property:

If $x(n)$ is one sided sequence

$$i) x(n-m) \xleftrightarrow{Z} z^{-m} \left[x(z) + \sum_{k=1}^m x(-k) z^k \right]$$

$$ii) x(n+m) \xleftrightarrow{Z} z^m \left[x(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right]$$

Proof:

$$Z\text{-transform of } x(n-m) = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) z^m z^{-m} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) z^m z^{-(n-m)}$$

$$= z^m \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)}$$

$$\text{Put } l = n-m \Rightarrow$$

$$\therefore = z^m \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$= z^m \left[\sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-m}^{-1} x(l) z^{-l} \right]$$

Put $l = -k$

$$\therefore = z^m \left[x(z) + \sum_{k=1}^m x(-k) z^k \right]$$

Put $l = -k$

$$\therefore = z^m \left[x(z) + \sum_{k=1}^m x(-k) z^k \right]$$

$$= z^m \left[x(z) + \sum_{k=1}^m x(-k) z^k \right]$$

$$x(n+m) \xleftrightarrow{z} z^m \left[x(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right]$$

$$x(n+m) \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} x(n+m) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n+m) z^{-(n+m)} \cdot z^m$$

$$= z^m \sum_{n=-\infty}^{\infty} x(n+m) z^{-(n+m)}$$

Put $l = n+m$

$$\therefore = z^m \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$= z^m \left[\sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=0}^{m-1} x(l) z^{-l} \right]$$

$$= z^m \left[x(z) - \sum_{l=0}^{m-1} x(l) z^{-l} \right]$$

$$\text{Put } = z^m \left[x(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right]$$

Qn. Find the z-transform of the following sequences and Roc using the properties of z-transform.

1) $x(n) = \delta(n-n_0)$.

$$\delta(n) \xleftrightarrow{Z} 1$$

By applying time shifting property we get

$$\delta(n-n_0) \xleftrightarrow{Z} \underline{\underline{z^{-n_0}}} \quad \text{Roc: } |z| > 0$$

2) $x(n) = u(n-n_0)$.

$$u(n) \xleftrightarrow{Z} \frac{z}{z-1}$$

By applying time shifting property

$$u(n-n_0) \xleftrightarrow{Z} z^{-n_0} \frac{z}{z-1} = \frac{z^{-(n_0-1)}}{z-1} \quad \text{Roc: } 1 < |z| < \infty$$

3) $x(n) = a^{n+1} u(n+1)$

$$a^n u(n) \xleftrightarrow{Z} \frac{z}{z-a}$$

$$a^{n+1} u(n+1) \xleftrightarrow{Z} \frac{z^{-1} z}{z-a} = \frac{z^2}{z-a} \quad \text{Roc: } |a| < |z| < \infty$$

4) $x(n) = a^{n-1} u(n-1)$

$$a^n u(n) \xleftrightarrow{Z} \frac{z}{z-a}$$

$$a^{n-1} u(n-1) \xleftrightarrow{Z} \frac{z^{-1} z}{z-a} = \frac{1}{z-a} \quad \text{Roc: } |a| < |z| < \infty$$

5) $x(n) = (\frac{1}{2})^n u(n)$

$$u(n) \xleftrightarrow{Z} \frac{z}{z-1}$$

$$u(n) \xleftrightarrow{Z} \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$$

By using multiplication by exponential seq. property,

$$x(n) = a^n u(n) \xleftrightarrow{Z} x(z) = \frac{1}{1-az}$$

$$(\frac{1}{2})^n u(n) \xleftrightarrow{Z} \frac{1}{1-(\frac{1}{2})z} = \frac{1}{1-2z} \quad \text{Roc: } |2||z| < 1$$

$$\text{Roc: } |z| < \frac{1}{2}$$

$$6) x(n) = n u(n).$$

$$u(n) \xleftrightarrow{Z} \frac{Z}{Z-1}$$

By using multiplication by 'n' property.

$$n u(n) \xleftrightarrow{Z} -Z \frac{d}{dz} \frac{Z}{Z-1} = -Z \frac{(Z-1) \cdot 1 - Z}{(Z-1)^2} \\ = \frac{Z}{(Z-1)^2}.$$

$$\boxed{n u(n) \xleftrightarrow{Z} \frac{Z}{(Z-1)^2}}$$

$$7) \text{ Show that } u(n) * u(n-1) = n u(n)$$

$$u(n) \xleftrightarrow{Z} \frac{Z}{Z-1} \quad u(n-1) \xleftrightarrow{Z} \frac{1}{Z-1}$$

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$$

$$\therefore u(n) * u(n-1) \xleftrightarrow{Z} \frac{Z}{Z-1} \cdot \frac{1}{Z-1} = \frac{Z}{(Z-1)^2}.$$

$$\text{W.K.T } n u(n) \xleftrightarrow{Z} \frac{Z}{(Z-1)^2}.$$

$$\therefore u(n) * u(n-1) = n u(n).$$

$$\textcircled{2} x(n) = \underbrace{n \left(-\frac{1}{4}\right)^n u(n)}_{x_1(n)} * \underbrace{\left(-\frac{1}{6}\right)^{-n} u(-n)}_{x_2(n)}.$$

$$\text{By using multiplication property } \left(-\frac{1}{4}\right)^n u(n) \xleftrightarrow{Z} \frac{Z}{Z + \frac{1}{4}}.$$

$$x_1(n) = n \left(-\frac{1}{4}\right)^n u(n) \xleftrightarrow{Z} -Z \frac{d}{dz} \frac{Z}{Z + \frac{1}{4}} \quad x_1(Z) =$$

$$\underline{x_1(Z)} = -Z \frac{(Z + \frac{1}{4}) - Z}{(Z + \frac{1}{4})^2} = \frac{-\frac{1}{4}Z}{(Z + \frac{1}{4})^2}. \quad \text{ROC: } |Z| > \frac{1}{4}$$

$$\left(-\frac{1}{6}\right)^{-n} u(-n) \xleftrightarrow{Z} \frac{Z}{Z + \frac{1}{6}}$$

By using time reversal property

$$\left(-\frac{1}{6}\right)^{-n} u(-n) \xleftrightarrow{Z} \frac{Z^{-1}}{Z^{-1} + \frac{1}{6}} = \frac{1}{1 + \frac{1}{6}Z} = \frac{6}{Z + 6}. \quad \text{ROC: } |Z| < 6$$

$$X(z) = \frac{-\frac{1}{4}z}{(z+\frac{1}{4})^2} \cdot \frac{+6}{(z+6)}$$

$$4(z^2 - \frac{1}{4}z - z + \frac{1}{4})$$

$$= \frac{-1.5z}{(z+\frac{1}{4})^2(z+6)} //$$

$$ROC: \frac{1}{4} < |z| < 6$$

Qn. Find the initial and final values of the following function.

$$x(z) = \frac{z}{4z^2 - 5z + 1}$$

$$ROC: |z| > 1$$

$$\text{initial value ; } x(0) = \lim_{z \rightarrow \infty} x(z)$$

$$= \lim_{z \rightarrow \infty} \frac{z}{4z^2 - 5z + 1}$$

$$= \lim_{z \rightarrow \infty} \frac{z}{z^2(4 - \frac{5}{z} + \frac{1}{z^2})}$$

$$\frac{1 - \frac{1}{4}}{\frac{4 - 1}{4}}$$

$$= 0$$

$$\text{Final value : } x(\infty) = \lim_{z \rightarrow 1} (z-1)x(z)$$

$$= \lim_{z \rightarrow 1} (z-1) \frac{z}{4z^2 - 5z + 1} = \lim_{z \rightarrow 1} \frac{z}{4(z-1)(z-\frac{1}{4})}$$

$$= \frac{1}{4(1-\frac{1}{4})}$$

$$= \frac{1}{4 \times \frac{3}{4}} = \frac{1}{3} //$$

Qn. Find $x(\infty)$ if $X(z) = \frac{z+1}{(z-0.6)^2}$

$$x(\infty) = \lim_{z \rightarrow 1} (z-1) X(z)$$

$$= \lim_{z \rightarrow 1} (z-1) \frac{(z+1)}{(z-0.6)^2} = 0$$

Qn. Find $x(\infty)$ if $X(z) = \frac{z+2}{4(z-1)(z+0.7)}$

$$x(\infty) = \lim_{z \rightarrow 1} (z-1) \frac{(z+2)}{4(z-1)(z+0.7)}$$

$$= \frac{3}{4 \times 1.7} = \frac{3}{6.8} //$$

Inverse System

The inverse z-transform of $X(z)$ is defined as

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz.$$

Inverse z-transform:

There are four methods that are given used for the evaluation of inverse z-transform.

- Long division method (power series method).
- partial fraction method
- Residue method
- Convolution method.

$$X(z) = \frac{N(z)}{D(z)}$$

* For getting right sided seq., the $N(z)$ and $D(z)$ must be put in descending power of z . before performing long division.

1) Long division method:

Using long division method, find the z -transform

$$X(z) = \frac{z}{z^2 - 3z + 1}$$

$$ROC: |z| > 1$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \rightarrow \textcircled{1}$$

Since $ROC: |z| > 1$, $x(n)$ must be a right sided seq.

$$\begin{array}{r} \frac{1}{2} z^{-1} + \frac{3}{4} z^{-2} + \frac{7}{8} z^{-3} + \dots \\ 2z^2 - 3z + 1 \overline{) \quad z} \\ \underline{(-) z \quad (+) \frac{3}{2} \quad (-) \frac{1}{2} z^{-1}} \\ \frac{3}{2} - \frac{1}{2} z^{-1} \\ \underline{(-) \frac{3}{2} \quad (+) \frac{9}{4} z^{-1} \quad (-) \frac{3}{4} z^{-2}} \\ \frac{7}{4} z^{-1} - \frac{3}{4} z^{-2} \\ \underline{(-) \frac{7}{4} z^{-1} \quad (+) \frac{21}{8} z^{-2} \quad (-) \frac{7}{8} z^{-3}} \\ \vdots \end{array}$$

$X(z) =$ Quotient of long division

$$= \frac{1}{2} z^{-1} + \frac{3}{4} z^{-2} + \frac{7}{8} z^{-3} + \dots \rightarrow \textcircled{2}$$

Comparing $\textcircled{1}$ and $\textcircled{2}$.

$$x(n) = \left(0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots \right)$$

For getting left sided seq., the $N(z)$ and $D(z)$ must be put in ascending power of z before performing long division.

Qn. using long division method, find the inverse z-transform of $X(z) = \frac{z}{2z^2 - 3z + 1}$, ROC: $|z| < \frac{1}{2}$.

Since ROC: $|z| < \frac{1}{2}$, $x(n)$ must be a left sided seq. For getting a left sided seq. the $N(z)$ & $D(z)$ must be put in ascending power of z .

$$X(z) = \sum_{n=-\infty}^0 x(n) z^{-n} = \dots + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) \rightarrow \textcircled{1}$$

$$\begin{array}{r}
 \overline{z+3z^2+7z^3+15z^4+\dots} \\
1-3z+2z^2 \quad \left| \begin{array}{l} z \\ (-)z^{(+)}-3z^{2(+)}+2z^3 \end{array} \right. \\
\hline
 3z^2-2z^3 \\
 (-)3z^{2(+)}-9z^{3(+)}+6z^4 \\
\hline
 7z^3-6z^4 \\
 (-)7z^{3(+)}-21z^{4(+)}+14z^5 \\
\hline
 15z^4-14z^5 \\
\vdots
\end{array}$$

$X(z) =$ Quotient of long division.

$$= \dots + 15z^4 + 7z^3 + 3z^2 + z \rightarrow \textcircled{2}$$

Comparing $\textcircled{1}$ and $\textcircled{2}$

$$x(n) = (\dots, 15, 7, 3, 1, 0)$$

↑

Qn. By using long division method determine the inverse z-transform of $X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$ when a) $x(n)$ is causal and b) $x(n)$ is anticausal.

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}} = \frac{z^2+z}{z^2-2z+1} = \frac{N(z)}{D(z)}$$

For getting a) $x(n)$ is causal causal seq, $N(z)$ & $D(z)$ must be put in descending power of z .

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \rightarrow \textcircled{1}$$

$$\begin{array}{r} 1+4z^{-1}+7z^{-2}+10z^{-3}+\dots \\ z^2-2z+1 \overline{) \quad \quad \quad} \\ \underline{z^2+2z} \\ (-) 4z-1 \\ \underline{(-) 4z+8+4z^{-1}} \\ 7-4z^{-1} \\ \underline{(-) 7+14z^{-1}+7z^{-2}} \\ 10z^{-1}-7z^{-2} \\ \vdots \end{array}$$

$X(z)$: Quotient of long division.

$$= 1+4z^{-1}+7z^{-2}+10z^{-3}+\dots \rightarrow \textcircled{2}$$

Comparing $\textcircled{1}$ and $\textcircled{2}$

$$x(n): (1, 4, 7, 10, \dots)$$

↑

For getting anti-causal seq., the NCZ) and DCZ) must be put in ascending power of z .

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \dots x(-4) z^4 + x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0)$$

→ ①

$$\begin{array}{r}
 1 - 2z + z^2 \quad \overline{) \quad 2z + 5z^2 + 8z^3 + 11z^4 + \dots} \\
 \underline{(-) 2z \quad (-) 4z^2 \quad (+) 2z^3} \\
 5z^2 - 2z^3 \\
 \underline{(-) 5z^2 \quad (+) 10z^3 \quad (-) 5z^4} \\
 8z^3 - 5z^4 \\
 \underline{(-) 8z^3 \quad (+) 16z^4 \quad (-) 8z^5} \\
 11z^4 - 8z^5
 \end{array}$$

$X(Z)$: Quotient of long division.

$$= \dots + 11z^4 + 8z^3 + 5z^2 + 2z \rightarrow \textcircled{1}$$

Comparing ① and ②

$$x(n) = (\dots, 11, 8, 5, 2, 0)$$

W.W.

using long division method find the inv. z-transform of

$X(z) = \frac{z+1}{z^2-8z+2}$ when a) $x(n)$ is causal b) $x(n)$ is anti-causal

Ans: a) $x(n) = (0, 1, 4, 10, 22)$

b) $x(n) = (\dots, 1.81, 1.62, 1.25, 0.5)$
 $\uparrow$

1) Partial fraction method:

$\frac{X(z)}{z}$	$x(n)$	
1) $\frac{z}{z-a}$	$a^n u(n)$	$ z > a \leftarrow \text{causal}$ if $x(n) \geq 0$
2) $\frac{z}{z-a}$	$-a^n u(-n-1)$	$ z < a \leftarrow \text{anti-causal}$ if $x(n) \leq 0$
3) $\frac{z}{z-1}$	$u(n)$	$ z > 1$
4) $\frac{z}{(z-a)^2}$	$n a^{n-1} u(n)$	
5) $\frac{z}{(z-a)^3}$	$\frac{n(n-1)}{2!} a^{n-2} u(n)$	

Qn. Find the inverse z-transform of $X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$
 ROC: $|z| > 2$

1)

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} = \frac{z(z - \frac{1}{3})}{(z-1)(z+2)}$$

2) Find $\frac{X(z)}{z}$.

$$\frac{X(z)}{z} = \frac{(z - \frac{1}{3})}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$z - \frac{1}{3} = A(z+2) + B(z-1)$$

Put $z=1 \Rightarrow 1 - \frac{1}{3} = 3A \Rightarrow \frac{2}{3} = 3A \Rightarrow A = \frac{2}{9}$

Put $z=-2 \Rightarrow -2 - \frac{1}{3} = -3B \Rightarrow -\frac{7}{3} = -3B \Rightarrow B = \frac{7}{9}$

$$\therefore \frac{X(z)}{z} = \frac{2}{9} \frac{1}{z-1} + \frac{7}{9} \frac{1}{z+2}$$

$$X(z) = \frac{2}{9} \frac{z}{z-1} + \frac{7}{9} \frac{z}{z+2}$$

Taking inverse z-transform.

$$x(n) = \frac{2}{9} u(n) + \frac{7}{9} (-2)^n u(n) //$$

Qn. Find the inverse z-transform of $X(z) = \frac{7z-23}{(z-3)(z-4)}$. ROC: $|z| > 4$

$$X(z) = \frac{7z-23}{(z-3)(z-4)}$$

$$\frac{X(z)}{z} = \frac{7z-23}{z(z-3)(z-4)} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$7z-23 = A(z-3)(z-4) + Bz(z-4) + Cz(z-3)$$

Put $z=0 \Rightarrow A = -\frac{23}{12}$, Put $z=3 \Rightarrow B = \frac{2}{3}$, Put $z=4 \Rightarrow C = \frac{5}{4}$

$$\frac{X(z)}{z} = -\frac{23}{12} \frac{1}{z} + \frac{2}{3} \frac{1}{z-3} + \frac{5}{4} \frac{1}{z-4}$$

$$X(z) = -\frac{23}{12} + \frac{2}{3} \frac{z}{z-3} + \frac{5}{4} \frac{z}{z-4}$$

Taking inverse z-transform.

$$x(n) = -\frac{23}{12} \delta(n) + \frac{2}{3} 3^n u(n) + \frac{5}{4} 4^n u(n)$$

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Qn. Determine the causal signal $x(n]$ having the z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

$$\frac{1 - 2z^{-1} + z^{-2}}{z^2 - 2z + 1}$$

$$X(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

~~$$z^2 = A(z-1)(z-1)^2 + B(z+1)(z-1)^2 + C(z+1)(z-1)$$~~

~~Put $z=1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$~~

$$\frac{z^2}{(z+1)(z-1)^2} = \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}$$

Put $z=1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

Put $z=-1 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

eqn coeff. of $z^2 \Rightarrow 1 = A+B \Rightarrow B = 1-A = 1-\frac{1}{4} = \frac{3}{4}$

$$\therefore \frac{X(z)}{z} = \frac{18}{4} \frac{1}{(z+1)} + \frac{3}{4} \frac{1}{(z-1)} + \frac{1}{2} \frac{1}{(z-1)^2}$$

$$X(z) = \frac{1}{4} \frac{z}{(z+1)} + \frac{3}{4} \frac{z}{(z-1)} + \frac{1}{2} \frac{z}{(z-1)^2}$$

Taking inverse z-transform.

$$\begin{aligned} x(n) &= \frac{1}{4} (-1)^n u(n) + \frac{3}{4} u(n) + \frac{1}{2} n (-1)^{n-1} u(n) \\ &= \frac{1}{4} (-1)^n u(n) + \frac{3}{4} u(n) + \frac{1}{2} n u(n) \end{aligned}$$

Ans. using partial fraction method find the inverse z-transform of $x(z) = \frac{z(z^2+z-30)}{(z-2)(z-4)^3}$, ROC $|z| > 4$

$$x(z) = \frac{z(z-5)(z+6)}{(z-2)(z-4)^3}$$

$$\frac{x(z)}{z} = \frac{(z-5)(z+6)}{(z-2)(z-4)^3} = \frac{A}{z-2} + \frac{B}{z-4} + \frac{C}{(z-4)^2} + \frac{D}{(z-4)^3}$$

Put $z=2 \Rightarrow A = 3$, Put $z=4 \Rightarrow C = -5$, ~~put~~

Compare the coeff. of $z^2 \Rightarrow C - 10B = 37$

compare the coeff. of $z \Rightarrow B = 6C - 32 \Rightarrow 138$

$C = 7$ and $B = -3$

$A_1 = A$
 $A_4 = B$
 $A_3 = C$
 $A_2 = D$

$$X(z) = 3 \frac{z}{z-2} - 3 \frac{z}{z-4} + 7 \frac{z}{(z-4)^2} - 5 \frac{z}{(z-4)^3}$$

Taking inverse

$$\begin{aligned} x(n) &= 3 \cdot 2^n u(n) - 3 \cdot 4^n u(n) + 7n \cdot 4^{n-1} u(n) - 5 \frac{n(n-1)}{2} 4^{n-2} u(n) \\ &= 3 \cdot 2^n u(n) - 3 \cdot 4^n u(n) + \frac{7}{4} n \cdot 4^n u(n) - \frac{5}{32} n(n-1) 4^n u(n) \end{aligned}$$

Determine the inverse Z-transform of $X(z) = \frac{5z^{-1}}{(1-2z^{-1})(1-3z^{-1})}$ for all possible ROCs.

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(1-3z^{-1})} = \frac{5z}{(z-2)(z-3)}$$

$$\frac{X(z)}{z} = \frac{5}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$5 = A(z-3) + B(z-2)$$

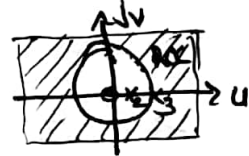
$$\text{Put } z=2 \Rightarrow 5 = -A \Rightarrow A = -5, \quad \text{Put } z=3 \Rightarrow 5 = B \Rightarrow B = 5$$

$$\therefore \frac{X(z)}{z} = -5 \frac{1}{z-2} + 5 \frac{1}{z-3}$$

$$X(z) = -5 \frac{z}{z-2} + 5 \frac{z}{z-3} \rightarrow \textcircled{1}$$

When the ROC is $|z| > 3$, then $x(n)$ is causal and all the two terms in eqn (1) are causal terms.

$$\therefore x(n) = -5 \cdot 2^n u(n) + 5 \cdot 3^n u(n) //$$



When the ROC is $|z| < 2$, then the signal $x(n)$ is anticausal and all the two terms in eqn (1) are anticausal terms.

$$\therefore x(n) = -5 \cdot (-1) 2^n u(-n-1) + 5 \cdot (-1) 3^n u(-n-1)$$

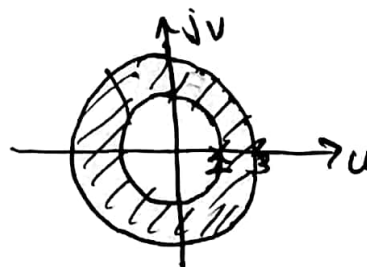
$$= 5 \cdot 2^n u(-n-1) - 5 \cdot 3^n u(-n-1)$$



When the ROC is $2 < |z| < 3$, then the signal is two-sided.

The pole $z=2$ provides causal term and the pole $z=3$ provides the anticausal term.

$$\therefore x(n) = -5 \cdot 2^n u(n) - 5 \cdot 3^n u(-n-1)$$



Ques. Qn

Qn. Find the inverse z-transform of $X(z) = \frac{z}{8z^2 - 4z + 1}$ for all possible ROCs.

$$X(z) = \frac{z}{(3z^2 - 4z + 1)} = \frac{z}{(z-1)(z-\frac{1}{3})}$$

$$1 - \frac{1}{3} = \frac{2}{3}$$

$$\frac{1}{3} - 1 = -\frac{2}{3}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-\frac{1}{3})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$1 = A(z-\frac{1}{3}) + B(z-1)$$

$$\text{Put } z=1 \Rightarrow 1 = (1-\frac{1}{3})A \Rightarrow 1 = \frac{2}{3}A \Rightarrow A = \frac{3}{2}$$

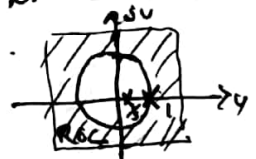
$$\text{Put } z=\frac{1}{3} \Rightarrow 1 = B(\frac{1}{3}-1) \Rightarrow 1 = -\frac{2}{3}B \Rightarrow B = -\frac{3}{2}$$

$$\therefore \frac{X(z)}{z} = \frac{3}{2} \frac{1}{z-1} - \frac{3}{2} \frac{1}{z-\frac{1}{3}}$$

$$X(z) = \frac{3}{2} \frac{z}{z-1} - \frac{3}{2} \frac{z}{z-\frac{1}{3}}$$

When ROC is $|z| > 1$, then the s/l $x(n)$ is causal and all the two terms are causal terms.

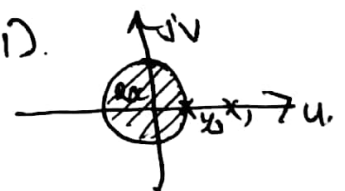
$$\therefore x(n) = \frac{3}{2} u(n) - \frac{3}{2} (\frac{1}{3})^n u(n)$$



When ROC is $|z| < \frac{1}{3}$, then the s/l $x(n)$ is anticausal and all the two terms are anticausal terms.

$$\therefore x(n) = \frac{3}{2} - (1)^n u(-n-1) - \frac{3}{2} (\frac{1}{3})^n u(-n-1)$$

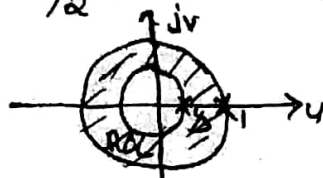
$$= -\frac{3}{2} u(-n-1) + \frac{3}{2} (\frac{1}{3})^n u(-n-1)$$



When ROC is $\frac{1}{3} < |z| < 1$, then the s/l $x(n)$ is two sided. The pole $z = \frac{1}{3}$ provides causal term and the pole $z = 1$ provides the anticausal term.

$$\therefore x(n) = \frac{3}{2} - (1)^n u(-n-1) - \frac{3}{2} (\frac{1}{3})^n u(n)$$

$$= -\frac{3}{2} u(-n-1) - \frac{3}{2} (\frac{1}{3})^n u(n)$$



Residue method:

$$x(n) = \frac{1}{2\pi j} \oint_C x(z) z^{n-1} dz = \sum \text{residues of } x(z) z^{n-1} \text{ at poles within } C.$$

Cauchy residue theorem:

Let $f(z)$ be a function of the complex variable z and C be a closed path in the z -plane. If the derivative $\frac{d}{dz} f(z)$ exists on and inside the C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz = f(z) \Big|_{z=z_0} = f(z_0).$$

If the $(k+1)^{\text{th}}$ order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} f(z) \Big|_{z=z_0}$$

Qn. By using residue method, find the inverse z -transform

of $x(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$, ROC: $|z| > 2$

$$x(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{z(z+3)}{z^2+3z+2} = \frac{z(z+3)}{(z+1)(z+2)}$$

$$x(n) = \sum \text{residues of } x(z) z^{n-1} \text{ at poles within } C.$$

$$= \sum \text{residues of } \frac{z(z+3)}{(z+1)(z+2)} z^{n-1} \text{ at poles within } C.$$

$$= \sum \text{residues of } \frac{(z+3)z^n}{(z+1)(z+2)} \text{ at poles } z = -1 \text{ and } z = -2 \text{ within } C.$$

$$= \text{residue of } \frac{(z+3)z^n}{(z+1)(z+2)} \text{ at pole } z = -1 + \text{residue of } \frac{(z+3)z^n}{(z+1)(z+2)} \text{ at pole } z = -2$$

$$= \frac{(z+3)z^n}{(z+1)(z+2)} \Big|_{z=-1} + \frac{(z+2)(z+3)z^n}{(z+1)(z+2)} \Big|_{z=-2}$$

$$= \frac{(z+3)z^n}{(z+2)} \Big|_{z=-1} + \frac{(z+3)z^n}{(z+1)} \Big|_{z=-2}$$

$$= \frac{2(-1)^n}{1} + \frac{1 \cdot (-2)^n}{-1} = \underline{\underline{[2(-1)^n - (-2)^n] u(n)}}$$

Qn. Find the inverse z-transform of $x(z) = \frac{z^2+z}{(z-1)(z-3)}$, ROC: $|z| > 3$.

$$x(z) = \frac{z^2+z}{(z-1)(z-3)} = \frac{z(z+1)}{(z-1)(z-3)}$$

$x(n) = \sum \text{residues of } x(z) z^{n-1} \text{ at poles within } C.$

$$= \sum \text{residues of } \frac{z(z+1)}{(z-1)(z-3)} z^{n-1} \text{ at poles within } C$$

$$= \sum \text{residue of } \frac{(z+1)z^n}{(z-1)(z-3)} \text{ at poles } z=1 \text{ \& } z=3 \text{ within } C$$

$$= \text{residue of } \frac{(z+1)z^n}{(z-1)(z-3)} \text{ at pole } z=1 + \text{residue of } \frac{(z+1)z^n}{(z-1)(z-3)} \text{ at pole } z=3$$

$$= \frac{(z+1)z^n}{(z-1)(z-3)} \Big|_{z=1} + \frac{(z+1)z^n}{(z-1)(z-3)} \Big|_{z=3}$$

$$= \frac{(z+1)z^n}{z-3} \Big|_{z=1} + \frac{(z+1)z^n}{z-1} \Big|_{z=3}$$

$$= \frac{2 \cdot 1^n}{-2} + \frac{4 \cdot 3^n}{2}$$

$$= -1^n + 2 \cdot 3^n$$

$$= \underline{\underline{[2 \cdot 3^n - 1^n] u(n)}}$$

Qn. Find $x(n)$ if $X(z) = \frac{2z^{-1}}{(1 - \frac{1}{4}z^{-1})^2}$; ROC $|z| > \frac{1}{4}$

~~Ans:~~ ~~Find~~ $X(z) z^{n-1}$

$$X(z) = \frac{2z^{-1}}{(1 - \frac{1}{4}z^{-1})^2} = \frac{2z}{(z - \frac{1}{4})^2}$$

$$= \frac{2z}{(z - \frac{1}{4})^2} z^{n-1} = \frac{2z^n}{(z - \frac{1}{4})^2} \rightarrow f(z).$$

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^k} dz = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} f(z) \Big|_{z=z_0}$$

$$\frac{1}{2\pi j} \oint_C \frac{2z^n}{(z - \frac{1}{4})^2} dz = \frac{d}{dz} f(z) \Big|_{z=\frac{1}{4}}$$

$$= \frac{d}{dz} 2z^n \Big|_{z=\frac{1}{4}} = 2n z^{n-1} \Big|_{z=\frac{1}{4}} = 2n \left(\frac{1}{4}\right)^{n-1} u(n)$$

Qn. Find $x(n)$ if $X(z) = \frac{z^2}{(z - \frac{1}{2})^2}$ ROC : $|z| > \frac{1}{4}$

$$X(z) z^{n-1} = \frac{z^2}{(z - \frac{1}{2})^2} z^{n-1} = \frac{z^{n+1}}{(z - \frac{1}{2})^2} \rightarrow f(z).$$

$$x(n) = \frac{1}{2\pi j} \oint \frac{z^{n+1}}{(z - \frac{1}{2})^2} dz = \frac{d}{dz} z^{n+1} \Big|_{z=\frac{1}{2}}$$

$$= (n+1) z^n \Big|_{z=\frac{1}{2}}$$

$$= (n+1) \left(\frac{1}{2}\right)^n u(n)$$

H.W 1) Find $x(n)$ if $X(z) = \frac{z^2 + z}{(z-1)(z-3)}$ ROC : $|z| > 3$.

Ans: $2(3)^n u(n) - u(n)$

2) Find $x(n)$ if $X(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{9}z^{-1}}$ ROC : $|z| > \frac{1}{3}$.

Ans: $\frac{1}{8} \left(\frac{1}{3}\right)^n + \frac{7}{8} \left(-\frac{1}{3}\right)^n u(n)$

~~Ans: Find x(n)~~

Q. Determine the inverse z-transform of $X(z) = \frac{z^{-1}}{(1-2z^{-1})(1-3z^{-1})}$
 ROC: $2 < |z| < 3$

$$X(z) = \frac{z}{(z-2)(z-3)}$$

from the ROC, we can see that $x(n)$ is a two sided seq.
 right sided $\rightarrow u(n)$
 left sided $\rightarrow u(-n-1)$

$$\begin{aligned} x(n) &= - \sum \text{residues of } X(z) z^{n-1} \text{ at pole } z=3 \\ &\quad + \sum \text{residue of } X(z) z^{n-1} \text{ at pole } z=2 \\ &= - \text{residue of } \frac{z^n}{(z-2)(z-3)} \text{ at pole } z=3 + \text{residue of } \frac{z^n}{(z-2)(z-3)} \text{ at pole } z=2 \\ &= - \left. \frac{(z/3) z^n}{(z-2)(z-3)} \right|_{z=3} + \left. \frac{(z/2) z^n}{(z-2)(z-3)} \right|_{z=2} \\ &= - \frac{3^n}{1} + \frac{2^n}{-1} \\ &\quad \text{left sided} \quad \text{right sided.} \\ &= - (3^n) u(-n-1) + (2^n) u(n) \end{aligned}$$

Convolution method:

Convolution property: $x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X(z) = X_1(z) X_2(z)$

from the property, we know that the convolution of $x_1(n)$ and $x_2(n)$ is the inverse z-transform of $X(z)$.

Thus $x(n)$ can be obtained by convolving $x_1(n)$ & $x_2(n)$.

Q. Find the inverse z-transform of $X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

ROC: $|z| > 2$ using convolution method.

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{z(z+3)}{z^2+3z+2}$$

$$X(z) = \frac{z}{(z+1)} \cdot \frac{z+3}{(z+2)}$$

$$\text{Let } x(z) = x_1(z) \cdot x_2(z)$$

$$\text{When } x_1(z) = \frac{z}{z+1} \Rightarrow x(n) = (-1)^n u(n)$$

$$\text{and } x_2(z) = \frac{z+3}{z+2} = \frac{z}{z+2} + z \frac{3}{z+2}$$

$$x_2(n) = (-2)^n u(n) + 3 \cdot (-2)^{n-1} u(n-1)$$

$$x(n) = x_1(n) * x_2(n)$$

$$= (-1)^n u(n) * [(-2)^n u(n) + 3(-2)^{n-1} u(n-1)]$$

$$= \underbrace{(-1)^n u(n) * (-2)^n u(n)}_{\text{"}} + (-1)^n u(n) * 3(-2)^{n-1} u(n-1)$$

$$= \sum_{k=-\infty}^{+\infty} (-1)^k u(k) (-2)^{n-k} u(n-k)$$

$$= \sum_{k=0}^n (-1)^k (-2)^n \cdot (-2)^{-k}$$

$$= \sum_{k=0}^n (0.5)^k (-2)^n = (-2)^n \sum_{k=0}^n (0.5)^k = (-2)^n \frac{1 - (0.5)^{n+1}}{1 - 0.5}$$

$$= (-2)^n \left(\frac{1 - 0.5^{n+1}}{0.5} \right) = (-2)^n (2 - 0.5^{n+1})$$

$$= 2(-2)^n - (-1)^{n+1}$$

$$(-1)^n u(n) * 3(-2)^{n-1} u(n-1) = 3 \sum_{k=-\infty}^{\infty} (-1)^k u(k) (-2)^{n-1-k} u(n-1-k)$$

$$= 3 \sum_{k=0}^{n-1} (-1)^k (-2)^{n-1-k} (-2)^{-k} = 3(-2)^{n-1} \sum_{k=0}^{n-1} (0.5)^k$$

$$= 3(-2)^{n-1} \left[\frac{1 - 0.5^n}{1 - 0.5} \right] = 3(-2)^{n-1} \left[\frac{1 - 0.5^n}{0.5} \right]$$

$$= 6(-2)^{n-1} - 6(-2)^{n-1} (0.5)^n$$

$$= 6(-2)^{n-1} \cdot \frac{1}{-2} - 6(-2)^{n-1} \cdot \frac{1}{-2} (0.5)^n$$

$$\begin{aligned}
 &= -3(-2)^n + 3(-1)^n. \\
 x(n) &= 2(-2)^n - (-1)^n + -3(-2)^n + 3(-1)^n. \\
 &= -(-2)^n + 2(-1)^n = \underbrace{[2(-1)^n - (-2)^n]}_{\text{}} \cdot u(n).
 \end{aligned}$$

Qn. Find the inverse Z-transform of $x(z) = \frac{z^2}{(z-2)(z-4)}$ using convolution method.

$$x(z) = \frac{z}{(z-2)} \cdot \frac{z}{(z-4)} = x_1(z) \cdot x_2(z).$$

$$x_1(z) = \frac{z}{z-2} \Rightarrow x_1(n) = 2^n u(n)$$

$$x_2(z) = \frac{z}{z-4} \Rightarrow x_2(n) = 4^n u(n)$$

$$\begin{aligned}
 n-k &\geq 0 \\
 n &\geq k \\
 k &\leq n.
 \end{aligned}$$

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k).$$

$$= \sum_{k=-\infty}^{\infty} 2^k u(k) 4^{n-k} u(n-k)$$

$$= \sum_{k=0}^n 2^k 4^{n-k} = \sum_{k=0}^n 2^k 4^n \cdot 4^{-k} = 4^n \sum_{k=0}^n 2^k \cdot \left(\frac{1}{4}\right)^k$$

$$= 4^n \sum_{k=0}^n \left(\frac{1}{2}\right)^k = 4^n \sum_{k=0}^n (0.5)^k = 4^n \left[\frac{1 - 0.5^{n+1}}{1 - 0.5} \right]$$

$$= 4^n \left(\frac{1 - 0.5^{n+1}}{0.5} \right) = 4^n [2 - 0.5^n]$$

$$= [2(4^n) - (2)^n] u(n).$$

Qn. Find the inverse z-transform of $X(z) = \frac{z}{(z-1)(z-\frac{1}{2})}$ using convolution property. Also verify the answer using partial fraction and residue method.

By convolution

$$X(z) = \frac{z}{z-1} \cdot \frac{1}{z-\frac{1}{2}} = x_1(z) \cdot x_2(z).$$

$$x_1(z) = \frac{z}{z-1} \Rightarrow x_1(n) = u(n)$$

$$x_2(z) = \frac{1}{z-\frac{1}{2}} = z^{-1} \frac{z}{z-\frac{1}{2}} \Rightarrow x_2(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1).$$

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-1-k} u(n-1-k).$$

$$= \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{n-1-k} = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right)^{-k}$$

$$= \left(\frac{1}{2}\right)^{n-1} \sum_{k=0}^{n-1} 2^k = \left(\frac{1}{2}\right)^{n-1} \left[\frac{1-2^n}{1-2} \right]$$

$$= \left(\frac{1}{2}\right)^{n-1} [2^n - 1] = 2^n \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n-1}$$

$$= [2 - 2\left(\frac{1}{2}\right)^n] u(n)$$

By partial fraction method: $\frac{X(z)}{z} = \frac{1}{(z-1)(z-\frac{1}{2})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$

$$1 = A(z-\frac{1}{2}) + B(z-1), \text{ Put } z=1 \Rightarrow \frac{1}{2}A = 1 \Rightarrow A=2.$$

$$\text{Put } z=\frac{1}{2} \Rightarrow -\frac{1}{2}B = 1 \Rightarrow B=-2.$$

$$X(z) = 2 \frac{z}{z-1} - 2 \frac{z}{z-\frac{1}{2}}$$

$$\text{Taking inv, } x(n) = 2u(n) - 2\left(\frac{1}{2}\right)^n u(n) = \underline{2 - 2\left(\frac{1}{2}\right)^n} u(n)$$

By residue method:

$$x(n) = \sum \text{residues of } \frac{z^n}{(z-1)(z-\frac{1}{2})} \text{ at poles within } C.$$

$$= \sum \text{residues of } \frac{z^n}{(z-1)(z-\frac{1}{2})} \text{ at poles } z=1 \text{ \& } z=\frac{1}{2} \text{ within } C.$$

$$= \text{residue of } \frac{z^n}{(z-1)(z-\frac{1}{2})} \text{ at pole } z=1 + \text{residue of } \frac{z^n}{(z-1)(z-\frac{1}{2})} \text{ at pole } z=\frac{1}{2}$$

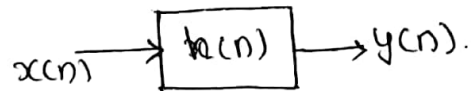
$$= \left. \frac{(z-1)z^n}{(z-\frac{1}{2})} \right|_{z=1} + \left. \frac{(z-\frac{1}{2})z^n}{(z-1)} \right|_{z=\frac{1}{2}} = \frac{1}{\frac{1}{2}} + \frac{\left(\frac{1}{2}\right)^n}{-\frac{1}{2}} = \underline{2 - 2\left(\frac{1}{2}\right)^n} u(n)$$

Analysis of LTI Systems:

The z-transform plays an important role in the analysis and design of discrete time LTI systems.

System function (transfer function) and impulse response:

Consider a discrete time LTI system having an impulse response $h(n)$ as shown in fig. given below.



Let us say it gives an output $y(n)$ for an input $x(n)$.

Then we have $y(n) = x(n) * h(n)$.

Taking z-transform on both sides.

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is called the system function or the transfer function of the LTI discrete time s/m and is defined as the ratio of the z-transform of the o/p sequence $y(n)$ to the z-transform of the i/p sequence $x(n)$, when the initial conditions are neglected.

The poles of the system are defined as the values of z for which the system function $H(z) = \infty$ and the zeros of the system are the values of z for which the system function $H(z) = 0$.

Relationship between transfer function and difference equation

Consider an n^{th} order LTI DT s/m described by the difference eqn.

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

Taking z-transform on both sides.

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

$$Y(z) \sum_{k=0}^N a_k z^{-k} = X(z) \sum_{k=0}^M b_k z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

When $\frac{Y(z)}{X(z)} = H(z)$ is called the system function or transfer function of the system. The frequency response of a system is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

Stability and causality:

The necessary and sufficient condition for a ~~causal~~ linear time invariant system to be BIBO stable is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

↓ The system is causal, $h(n) = 0$ for $n < 0$.

For a causal LTI s/m the condition for stability is

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

System function, $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$.

of a causal LTI s/m
magnitude $|H(z)| = \left| \sum_{n=0}^{\infty} h(n) z^{-n} \right| = \sum_{n=0}^{\infty} |h(n)| |z^{-n}|$

The evaluation of $|H(z)|$ on unit circle yields.

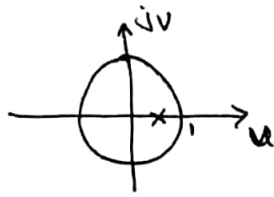
$$|H(z)| = \sum_{n=0}^{\infty} |h(n)| < \infty \quad (\because |z| = 1 \text{ for unit circle})$$

Therefore for a stable system, the ROC of a system function includes unit circle.

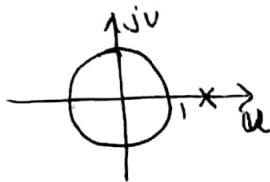
For a causal system, the ROC is exterior of the circle of radius 'r'. Since ROC cannot contain any pole of $H(z)$, a causal LTI system is BIBO stable.

show that for a stable s/m, the ROC of a s/m function includes the unit circle.

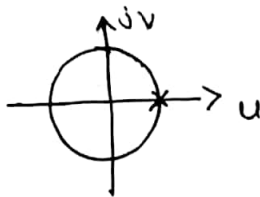
if and only if all the poles of $H(z)$ are inside the unit circle.



→ stable because a pole is inside the unit circle.



→ unstable because pole is outside the unit circle.



→ unstable because pole is in the unit circle.

Problems on system function

- 1) Find the s/m function of the 1st order difference equation
 $y(n) - 2y(n-1) = x(n) + x(n-1)$

Taking z-transform on both sides.

$$Y(z) - 2z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)[1 - 2z^{-1}] = X(z)[1 + z^{-1}]$$

System function $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 2z^{-1}}$

- 2) Given $x(n) = u(n)$ and $y(n) = 2^n u(n)$. Find the system function and impulse response.

$$X(z) = \frac{z}{z-1} \quad \text{and} \quad Y(z) = \frac{z}{z-2}$$

System function $H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-2} \times \frac{z-1}{z}$

$$H(z) = \frac{z-1}{z-2}$$

Impulse response, $h(n)$:

$$\frac{H(z)}{z} = \frac{z-1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2} \Rightarrow z-1 = A(z-2) + Bz$$

Put $z=0 \Rightarrow -1 = -2A \Rightarrow A = 1/2$, Put $z=2 \Rightarrow 1 = 2B$
 $B = 1/2$

$$\therefore \frac{H(z)}{z} = \frac{1}{2} \frac{1}{z} + \frac{1}{2} \frac{1}{z-2}$$

$$\therefore H(z) = \frac{1}{2} + \frac{1}{2} \frac{z}{z-2}$$

Taking inverse, $h(n) = \frac{1}{2} \delta(n) + \frac{1}{2} 2^n u(n)$

3) An LTI s/m is described by the following s/m function
 $H(z) = \frac{z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})}$. Find the s/m response to the i/p

$$x(n) = 4^{-(n+2)} u(n)$$

$$x(n) = 4^{-(n+2)} u(n) = 4^{-n} \cdot 4^{-2} u(n) = \left(\frac{1}{4}\right)^n u(n)$$

$$X(z) = \frac{1}{16} \frac{z}{z - \frac{1}{4}}$$

W.K.T $H(z) = \frac{Y(z)}{X(z)} \Rightarrow Y(z) = H(z) \cdot X(z)$

$$Y(z) = \frac{z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})} \times \frac{1}{16} \frac{z}{z - \frac{1}{4}}$$

$$\frac{Y(z)}{z} = \frac{1}{16} \frac{z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-1)} + \frac{B}{(z-\frac{1}{2})} + \frac{C}{z-\frac{1}{4}}$$

$$\frac{1}{16} (z + \frac{1}{2}) = A(z - \frac{1}{2})(z - \frac{1}{4}) + B(z-1)(z - \frac{1}{4}) + C(z-1)(z - \frac{1}{2})$$

Put $z = \frac{1}{2} \Rightarrow \frac{1}{16} = B\left(\frac{1}{2} - \frac{1}{4}\right) \Rightarrow \frac{1}{16} = -\frac{1}{8} B \Rightarrow B = -\frac{1}{2}$

Put $z = \frac{1}{4} \Rightarrow \frac{1}{16} \times \frac{3}{4} = C(-\frac{3}{4})(-\frac{1}{4}) \Rightarrow \frac{3}{16 \times 4} = \frac{3}{16} C \Rightarrow C = \frac{3}{16 \times 4} \times \frac{16}{3} = \frac{1}{4}$

Put $z = 1 \Rightarrow \frac{1}{16} \times \frac{3}{2} = A(\frac{1}{2})(\frac{3}{4}) \Rightarrow \frac{3}{16 \times 2} = \frac{3}{8} A \Rightarrow A = \frac{3}{16 \times 2} \times \frac{8}{3} = \frac{1}{4}$

$$\therefore \frac{Y(z)}{z} = \frac{1}{4} \frac{1}{(z-1)} - \frac{1}{2} \frac{1}{(z-\frac{1}{2})} + \frac{1}{4} \frac{1}{(z-\frac{1}{4})}$$

$$Y(z) = \frac{1}{4} \frac{z}{z-1} - \frac{1}{2} \frac{z}{z-\frac{1}{2}} + \frac{1}{4} \frac{z}{z-\frac{1}{4}}$$

Taking inverse z-transform

$$y(n) = \frac{1}{4} u(n) - \frac{1}{2} \left(\frac{1}{2}\right)^n u(n) + \frac{1}{4} \left(\frac{1}{4}\right)^n u(n)$$

System response

N.W

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4) The i/p to the system $x(n) = u(n-1) + (\frac{1}{2})^n u(n)$. The z-transform of the s/m o/p is $Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})}$. Determine the impulse response and o/p of the system.

$$X(z) = \frac{\frac{1}{2}z}{(1-z)(z-\frac{1}{2})}, \quad Y(z) = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} = \frac{-\frac{1}{2}z}{(z-\frac{1}{2})(z+1)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z}{(z-\frac{1}{2})(z+1)} \times \frac{(1-z)(z-\frac{1}{2})}{\frac{1}{2}z} = \frac{-(1-z)}{z+1} = \frac{z-1}{z+1}$$

impulse response $h(n)$:

$$\frac{H(z)}{z} = \frac{z-1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$$

$$\Rightarrow z-1 = A(z+1) + Bz$$

Put $z=0 \Rightarrow -1 = A$; put $z=-1 \Rightarrow -2 = -B \Rightarrow B=2$

$$\therefore \frac{H(z)}{z} = -\frac{1}{z} + 2 \frac{1}{z+1}$$

$$H(z) = -1 + 2 \frac{z}{z+1}$$

Taking inv. z-transform, impulse response, $h(n) = \underline{-\delta(n) + 2(-1)^n u(n)}$

o/p of the system $y(n)$:

$$\frac{Y(z)}{z} = \frac{-\frac{1}{2}}{(z-\frac{1}{2})(z+1)} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{z+1}$$

$$A = -\frac{1}{3} \text{ and } B = \frac{1}{3}$$

$$\frac{Y(z)}{z} = -\frac{1}{3} \frac{1}{z-\frac{1}{2}} + \frac{1}{3} \frac{1}{z+1}$$

$$Y(z) = -\frac{1}{3} \frac{z}{z-\frac{1}{2}} + \frac{1}{3} \frac{z}{z+1}$$

Taking inverse z-transform,

$$\text{o/p of the system, } y(n) = \underline{-\frac{1}{3}(\frac{1}{2})^n u(n) + \frac{1}{3}(-1)^n u(n)}$$

Qn. A causal system is represented by $H(z) = \frac{z+2}{2z^2-3z+4}$

find the difference eqn. and the frequency response of the system.

Given $H(z) = \frac{z+2}{2z^2-3z+4}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z+2}{2z^2-3z+4} = \frac{z^2(z^{-1}+2z^{-2})}{z^2(2-3z^{-1}+4z^{-2})}$$

$$Y(z)(2-3z^{-1}+4z^{-2}) = X(z)(z^{-1}+2z^{-2})$$

$$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

Taking inverse z-transform.

$$2y(n) - 3y(n-1] + 4y(n-2) = x(n-1] + 2x(n-2)$$

which is the required difference eqn.

Putting $z = e^{j\Omega}$ in $H(z)$, we get the frequency response $H(e^{j\Omega})$ of the system.

$$H(z) = \frac{z+2}{2z^2-3z+4}$$

$$\text{freq. resp. } H(e^{j\Omega}) = \frac{e^{j\Omega} + 2}{2e^{j2\Omega} - 3e^{j\Omega} + 4}$$

$$= \frac{2 + \cos\Omega + j\sin\Omega}{4 + (2\cos 2\Omega + j2\sin 2\Omega - 3\cos\Omega - 3j\sin\Omega)}$$

$$= \frac{2 + \cos\Omega + j\sin\Omega}{4 + (2\cos 2\Omega - 3\cos\Omega) + j(2\sin 2\Omega - 3\sin\Omega)}$$

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3. plot the pole-zero pattern and determine which of the following systems are stable:

a) $y(n) = y(n-1) - 0.8 y(n-2) + x(n) + x(n-2)$

b) $y(n) = 2y(n-1) - 0.8 y(n-2) + x(n) + 0.8 x(n-1)$

a) Given $y(n) = y(n-1) - 0.8 y(n-2) + x(n) + x(n-2)$

Taking z-transform.

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

$$Y(z) - z^{-1}Y(z) + 0.8z^{-2}Y(z) = X(z) + z^{-2}X(z)$$

$$Y(z) [1 - z^{-1} + 0.8z^{-2}] = X(z) [1 + z^{-2}]$$

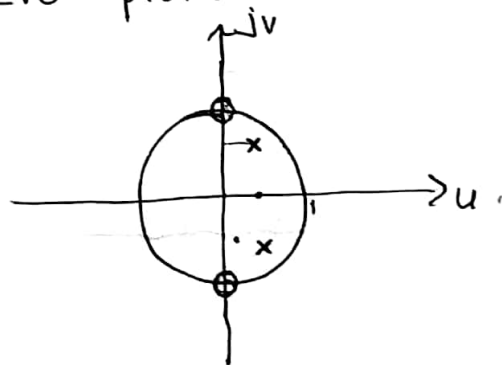
System function, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 - z^{-1} + 0.8z^{-2}}$

$$= \frac{z^2 + 1}{z^2 - z + 0.8} = \frac{(z+j)(z-j)}{(z-0.5-j0.74)(z-0.5+j0.74)}$$

Zeros of $H(z)$: $z = 1j$ and $z = -1j$

Poles of $H(z)$: $z = 0.5 + j0.74$ and $z = 0.5 - j0.74$

pole-zero plot:



All the poles are inside the unit circle. Hence the system is stable.

b) Given $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

$$Y(z) = 2z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + 0.8z^{-1}X(z)$$

$$Y(z) - 2z^{-1}Y(z) + 0.8z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z)$$

$$Y(z)[1 - 2z^{-1} + 0.8z^{-2}] = X(z)[1 + 0.8z^{-1}]$$

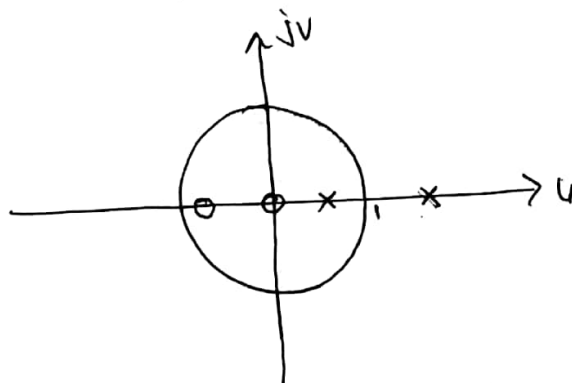
System function, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1}}{1 - 2z^{-1} + 0.8z^{-2}}$

$$H(z) = \frac{z^2 + 0.8z}{z^2 + 2z + 0.8} = \frac{z(z + 0.8)}{(z - 1.44)(z - 0.55)}$$

Zeros of $H(z)$: $z = 0$ and $z = -0.8$

Poles of $H(z)$: $z = 1.44$ and $z = 0.55$

Pole-zero plot:



One pole is outside the unit circle.

$\therefore$ The s/m is unstable.

Qn. Consider a causal discrete time LTI s/m with input $x(n) = (\frac{1}{3})^n u(n) - 2(\frac{1}{3})^{n-1} u(n-1)$ and output $y(n) = (\frac{1}{2})^n u(n)$.

Determine the transfer function ($H(z)$), impulse response ($h(n)$) and frequency response $H(e^{j\omega})$ of the system.

Given

$$x(n) = \underbrace{(\frac{1}{3})^n u(n)}_{\frac{z}{z - \frac{1}{3}}} - 2 \underbrace{(\frac{1}{3})^{n-1} u(n-1)}_{2z^{-1} \frac{z}{z - \frac{1}{3}}}$$

$$X(Z) = \frac{Z}{Z - \frac{1}{3}} - \frac{2}{Z - \frac{1}{3}} = \frac{Z-2}{Z - \frac{1}{3}}$$

Given $y(n) = (\frac{1}{2})^n u(n)$ $\therefore Y(Z) = \frac{Z}{Z - \frac{1}{2}}$

System function, $H(Z) = \frac{Y(Z)}{X(Z)} = \frac{\frac{Z}{Z - \frac{1}{2}} \times \frac{Z-2}{Z - \frac{1}{3}}}{\frac{Z(Z - \frac{1}{3})}{(Z - \frac{1}{2})(Z - \frac{1}{3})}}$

$$H(Z) = \frac{Z(Z-2)}{(Z - \frac{1}{2})(Z - \frac{1}{3})}$$

$$\frac{1}{2} - 2 = \frac{1-4}{2} = -\frac{3}{2}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$\frac{1}{3} - 2 = \frac{1-6}{3} = -\frac{5}{3}$$

$$\frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$$

Impulse response, $h(n)$:

$$\frac{H(Z)}{Z} = \frac{Z-2}{(Z - \frac{1}{2})(Z - \frac{1}{3})} = \frac{A}{(Z - \frac{1}{2})} + \frac{B}{(Z - \frac{1}{3})}$$

$$Z-2 = A(Z - \frac{1}{3}) + B(Z - \frac{1}{2})$$

Put $Z = \frac{1}{2} \Rightarrow -\frac{3}{2} = \frac{1}{6}A \Rightarrow A = -\frac{3}{2} \times 6 = -9$

Put $Z = \frac{1}{3} \Rightarrow -\frac{5}{3} = -\frac{1}{6}B \Rightarrow B = -\frac{5}{3} \times -6 = 10$

$$\therefore \frac{H(Z)}{Z} = -9 \frac{1}{Z - \frac{1}{2}} + 10 \frac{1}{Z - \frac{1}{3}}$$

$$H(Z) = -9 \frac{Z}{Z - \frac{1}{2}} + 10 \frac{Z}{Z - \frac{1}{3}}$$

Taking inverse z-transform

impulse response, $h(n) = -9 (\frac{1}{2})^n u(n) + 10 (\frac{1}{3})^n u(n)$

frequency response: $H(e^{j\Omega})$.

$$H(Z) = \frac{Z(Z-2)}{(Z - \frac{1}{2})(Z - \frac{1}{3})}$$

$$H(e^{j\Omega}) = \frac{e^{j\Omega}(e^{j\Omega} - 2)}{(e^{j\Omega} - \frac{1}{2})(e^{j\Omega} - \frac{1}{3})}$$

Qn. Determine the impulse response and step response of the causal system given below and discuss on stability. $y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

$$\text{given } y(n) - y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

$$Y(z) - z^{-1}Y(z) - 2z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

$$Y(z)[1 - z^{-1} - 2z^{-2}] = X(z)[z^{-1} + 2z^{-2}]$$

$$\text{System fun.}, H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + 2z^{-2}}{1 - z^{-1} - 2z^{-2}}$$

$$H(z) = \frac{z+2}{z^2 - z - 2} = \frac{z+2}{(z-2)(z+1)}$$

$$\text{Impulse response, } h(n): \quad \frac{H(z)}{z} = \frac{z+2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{(z+1)}$$

$$z+2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

$$\text{Put } z=0 \Rightarrow 2 = A(-2)(1) \Rightarrow 2 = -2A \Rightarrow A = -1$$

$$\text{Put } z=2 \Rightarrow 4 = 6B \Rightarrow B = 4/6 = 2/3$$

$$\text{Put } z=-1 \Rightarrow 1 = 3C \Rightarrow C = 1/3$$

$$\therefore \frac{H(z)}{z} = -\frac{1}{z} + \frac{2}{3} \frac{1}{z-2} + \frac{1}{3} \frac{1}{z+1}$$

* unstable

$$H(z) = -1 + \frac{2}{3} \frac{z}{z-2} + \frac{1}{3} \frac{z}{z+1}$$

Taking inverse

$$h(n) = -\delta(n) + \frac{2}{3} 2^n u(n) + \frac{1}{3} (-1)^n u(n)$$

$$\text{For step response, } x(n) = u(n) \Rightarrow X(z) = \frac{z}{z-1}$$

$$\text{output, } Y(z) = H(z) \cdot X(z) = \frac{z+2}{(z-2)(z+1)} \cdot \frac{z}{(z-1)}$$

$$\frac{Y(z)}{z} = \frac{z+2}{(z-2)(z+1)(z-1)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z-1} \quad \left| \begin{array}{l} A = 4/3 \\ B = 1/6 \\ C = -3/2 \end{array} \right.$$

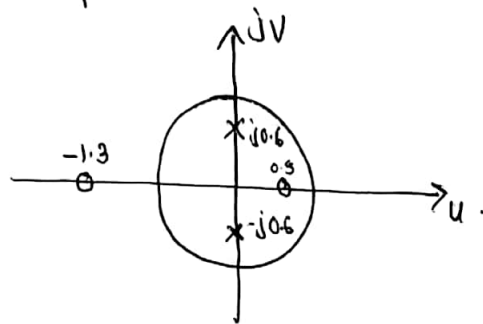
$$\therefore y(n) = \frac{4}{3} 2^n u(n) + \frac{1}{6} (-1)^n u(n) - \frac{3}{2} u(n) //$$

Determining the frequency response from poles & zeros:

Step (1) from the pole-zeros, write the system function $H(z)$

Step (2) Find $H(z) \big|_{z=e^{j\Omega}}$; we get $H(e^{j\Omega})$, is the frequency response.

Qn. Determine the frequency response from the following pole-zero plot.



Zeros : $z = -1.3$ and $z = 0.5$

poles : $z = j0.6$ and $z = -j0.6$.

$$H(z) = \frac{(z+1.3)(z-0.5)}{(z-j0.6)(z+j0.6)}$$

$$\text{frequency response : } H(e^{j\Omega}) = \frac{(e^{j\Omega}+1.3)(e^{j\Omega}-0.5)}{(e^{j\Omega}-j0.6)(e^{j\Omega}+j0.6)}$$

Qn. The zeros of $H(z)$ are $z=0$ and $z=-0.8$ and the poles of $H(z)$ are $z=1.4$ and $z=0.5$. Determine the frequency response of the s/m.

$$H(z) = \frac{z(z+0.8)}{(z-1.4)(z-0.5)}$$

$$\text{freq. resp, } H(e^{j\Omega}) = \frac{e^{j\Omega}(e^{j\Omega}+0.8)}{(e^{j\Omega}-1.4)(e^{j\Omega}-0.5)} //$$

Solution of LTI Systems described by the difference eqs:

$$\left. \begin{aligned} y(n-1) &\xleftrightarrow{z} z^{-1} y(z) + y(-1) \\ y(n-2) &\xleftrightarrow{z} z^{-2} y(z) + z^{-1} y(-1) + y(-2) \end{aligned} \right\} \text{shifting property.}$$

$y(-1), y(-2) \rightarrow$ initial conditions.

Steps: 1) For a given set of initial conditions, take the z-transform of both sides of the diff. eqn. to obtain the algebraic eqn. in $y(z)$.

2) Solve the algebraic eqn. for $y(z)$

3) Take the inverse z-transform.

Qn. Determine the step response of the system

$$y(n) - \frac{1}{2} y(n-1) = x(n) - \frac{1}{2} x(n-1). \text{ Assume the initial conditions as } y(-1) = 1$$

$$y(-1) = 1$$

$$y(n) - \frac{1}{2} y(n-1) = x(n) - \frac{1}{2} x(n-1)$$

For step response
 $x(n) = u(n)$
 $X(z) = \frac{z}{z-1}$

Taking z-transform.

$$Y(z) - \frac{1}{2} z^{-1} Y(z) + y(-1) = X(z) - \frac{1}{2} z^{-1} X(z) + x(-1)$$

$$Y(z) - \frac{1}{2} [z^{-1} Y(z) + y(-1)] = X(z) - \frac{1}{2} z^{-1} X(z)$$

$$Y(z) [1 - \frac{1}{2} z^{-1}] + \frac{1}{2} = [1 - \frac{1}{2} z^{-1}] X(z)$$

$$Y(z) = \frac{\frac{1}{2} [1 - \frac{1}{2} z^{-1}]}{[1 - \frac{1}{2} z^{-1}]} \frac{z}{z-1}$$

$$= \frac{1}{2} \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{z}{z-1}$$

$$Y(z) = \frac{1}{2} \frac{z}{z - \frac{1}{2}} + \frac{z}{z-1}$$

Taking inverse z-transform

$$y(n) = \frac{1}{2} \left(\frac{1}{2} \right)^n u(n) + u(n)$$

Qn. Determine the response of the s/m given by the difference eqn. $y(n) - 0.5y(n-1) = x(n)$, when the input is $x(n) = 5^n u(n)$ and the initial condition is $y(-1) = 2$.

Given $x(n) = 5^n u(n) \Rightarrow X(Z) = \frac{Z}{Z-5}$

$y(n) - 0.5y(n-1) = x(n)$

Taking z-transform.

$Y(Z) - 0.5[Z^{-1}Y(Z) + y(-1)] = X(Z)$

$Y(Z) - 0.5[Z^{-1}Y(Z) + 2] = X(Z)$

$Y(Z) - 0.5Z^{-1}Y(Z) - 1 = X(Z)$

$Y(Z)[1 - 0.5Z^{-1}] + 1 = X(Z)$

$Y(Z)[1 - 0.5Z^{-1}] = X(Z) - 1$

$Y(Z)[1 - 0.5Z^{-1}] = \frac{Z}{Z-5} - 1$

$Y(Z) = \frac{\frac{Z}{Z-5} - 1}{1 - 0.5Z^{-1}} + \frac{1}{1 - 0.5Z^{-1}}$

$= \frac{Z}{(Z-5)(1-0.5Z^{-1})} + \frac{1}{1-0.5Z^{-1}}$

$= \underbrace{\frac{Z^2}{(Z-5)(Z-0.5)}}_{Y_1(Z)} + \underbrace{\frac{Z}{Z-0.5}}_{Y_2(Z) \Rightarrow y_2(n) = 0.5^n u(n)}$

To find

$y(n) = y_1(n) + y_2(n)$

To find $y_1(n)$:

$\frac{Y_1(Z)}{Z} = \frac{Z}{(Z-5)(Z-0.5)} = \frac{A}{(Z-5)} + \frac{B}{(Z-0.5)}$

$A = 10/9$

$B = -1/9$

$\frac{Y_1(Z)}{Z} = \frac{10}{9} \frac{1}{(Z-5)} - \frac{1}{9} \frac{1}{(Z-0.5)}$

$1 - \frac{1}{9} \frac{1}{Z-0.5}$

$Y_1(Z) = \frac{10}{9} \frac{Z}{Z-5} - \frac{1}{9} \frac{Z}{Z-0.5}$

Taking inv. z-transform

$y_1(n) = 10/9 5^n u(n) - 1/9 (0.5)^n u(n)$

$y(n) = y_1(n) + y_2(n) = 10/9 5^n u(n) - \frac{1}{9} (0.5)^n u(n) + 0.5^n u(n)$
 $= 10/9 5^n u(n) + 8/9 0.5^n u(n)$

Zero input and zero state response:

The response due to the initial conditions alone (in the absence of i/p, $x(n)=0$) is called zero i/p response. The response due to the i/p alone (assuming that all the initial conditions are zero) is called zero state response.

Total response = zero i/p response + zero state response.

1. ~~Given~~ $y(n) + 5y(n-1) + 6y(n-2) = x(n-1) + 2x(n)$,
where $x(n) = u(n)$. The initial conditions are $y(-1)=1, y(-2)=0$
Find a) zero i/p response
b) zero state response
c) Total response.

a) $y(n) + 5y(n-1) + 6y(n-2) = x(n-1) + 2x(n)$.

Taking Z transform.

$$Y(z) + 5[z^{-1}Y(z) + y(-1)] + 6[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = [z^{-1}X(z) + X(-1)] + 2X(z).$$

a) zero i/p response:

$$Y(z) + 5[z^{-1}Y(z) + \underbrace{y(-1)}_{=1}] + 6[z^{-2}Y(z) + z^{-1}\underbrace{y(-1)}_{=1} + \underbrace{y(-2)}_{=0}] = 0$$

$$Y(z) + 5z^{-1}Y(z) + 5 + 6z^{-2}Y(z) + 6z^{-1} = 0.$$

$$Y(z)[1 + 5z^{-1} + 6z^{-2}] + 5 + 6z^{-1} = 0.$$

$$Y(z)[1 + 5z^{-1} + 6z^{-2}] = -5 - 6z^{-1}$$

$$Y(z) = \frac{-5 - z^{-1}}{1 + 5z^{-1} + 6z^{-2}} = \frac{z(-5z - 1)}{z^2 + 5z + 6} = \frac{z(-5z - 6)}{(z+2)(z+3)}$$

$$\frac{Y(z)}{z} = \frac{-5z - 6}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}.$$

~~Partial fraction~~ $A = 4, B = -9.$

$$\therefore \frac{Y(z)}{z} = 4 \frac{1}{z+2} - 9 \frac{1}{z+3}$$

$$Y(z) = 4 \frac{z}{z+2} - 9 \frac{z}{z+3}$$

Taking inverse

$$y(n) = 4(-2)^n u(n) - 9(-3)^n u(n)$$

b) Zero state response:

$$Y(z) + 5[z^{-1}Y(z) + 0] + 6[z^{-2}Y(z) + 0 + 0] = z^{-1}X(z) + 2X(z)$$

$$Y(z) + 5z^{-1}Y(z) + 6z^{-2}Y(z) = z^{-1}X(z) + 2X(z)$$

$$Y(z)[1 + 5z^{-1} + 6z^{-2}] = \cancel{X(z)} [z^{-1} + 2] X(z)$$

$$Y(z) = \frac{[z^{-1} + 2]}{[1 + 5z^{-1} + 6z^{-2}]} \cdot \frac{Z}{Z-1}$$

$$= \frac{Z(1 + 2Z)}{(Z+2)(Z+3)} \cdot \frac{Z}{(Z-1)}$$

$$\frac{Y(z)}{Z} = \frac{Z(1 + 2Z)}{(Z+2)(Z+3)(Z-1)} = \frac{A}{Z+2} + \frac{B}{Z+3} + \frac{C}{Z-1}$$

$$Z(1 + 2Z) = A(Z+3)(Z-1) + B(Z+2)(Z-1) + C(Z+2)(Z+3)$$

$$A = -2, \quad B = \frac{15}{4}, \quad C = \frac{1}{4}$$

$$\therefore \frac{Y(z)}{Z} = -2 \frac{1}{Z+2} + \frac{15}{4} \frac{1}{Z+3} + \frac{1}{4} \frac{1}{Z-1}$$

$$Y(z) = -2 \frac{Z}{Z+2} + \frac{15}{4} \frac{Z}{Z+3} + \frac{1}{4} \frac{Z}{Z-1}$$

$$\frac{15}{4} - 9 = \frac{15 - 36}{4} = \frac{-21}{4}$$

Taking inv. z-transform

$$y(n) = -2(-2)^n u(n) + \frac{15}{4}(-3)^n u(n) + \frac{1}{4}u(n)$$

Total response = Zero i/p response + Zero state response

$$= \frac{4}{4}(-2)^n u(n) - \frac{9}{4}(-3)^n u(n) - \frac{2(-2)^n u(n)}{4} + \frac{15}{4}(-3)^n u(n) + \frac{1}{4}u(n)$$

$$= 2(-2)^n u(n) - \frac{21}{4}(-3)^n u(n) + \frac{1}{4}u(n)$$

Fourier Representation for periodic discrete time signal.

→ Discrete time Fourier Series.

Definition of DTFS:

DTFS representation of a periodic sequence $x(n)$ is given by

$$x(n) = \sum_{k=-\infty}^{\infty} X(k) e^{jK\omega n} \rightarrow \textcircled{1} \text{ (Synthesis eqn)}$$

The inverse DTFS is

$$X(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x(n) e^{-jK\omega n} \rightarrow \textcircled{2} \text{ (Analysis eqn)}$$

$N \rightarrow$ Fundamental period

~~as fundamental frequency~~

$$k, n \in -(N/2-1) \text{ to } N/2 \rightarrow \text{if } N \text{ is even}$$

$$k, n \in -\frac{(N-1)}{2} \text{ to } \frac{(N-1)}{2} \rightarrow \text{if } N \text{ is odd}$$

In the above eqns. $\textcircled{1}$ and $\textcircled{2}$, $X(k)$ are known as DTFS coefficients. Eqn. $\textcircled{1}$ is known as Synthesis eqn and $\textcircled{2}$ is known as Analysis eqn.

Steps to find Fourier coefficients $X(k)$:

from the function:

- 1) Find the value of N and ω
- 2) Express the given function in exponential form
- 3) Express the above fun. in terms of ω , put it as eqn $\textcircled{1}$.

8) Express the above

4) Write the synthesis eqn.

$$x(n) = \sum_{k \in \langle N \rangle} x(k) e^{j k \Omega n}$$

5) Select the range of k

$$k \in -\frac{N-1}{2} \text{ to } \frac{N-1}{2} \rightarrow \text{if } N \text{ is even}$$

$$k \in -\frac{(N-1)}{2} \text{ to } \frac{(N-1)}{2} \rightarrow \text{if } N \text{ is odd}$$

6) Expand the synthesis eqn. by putting the values of k . Put it as eqn ②.

7) Comparing ① and ②, we get the Fourier coefficient $x(k)$

From the figure:

1) Find $N, \Omega = 2\pi/N$.

2) Select the range of n

3) Write the analysis eqn: $x(k) = \frac{1}{N} \sum_{n \in \langle N \rangle} x(n) e^{-j k \Omega n}$

4) Expand the above eqn. by putting the values of n

Qn. Determine the DFTS coefficients of $x(n) = 3 \cos(\pi/8 n)$

Step (1): Find the value Ω and N

$$\Omega = \pi/8$$

$$N \rightarrow \frac{2\pi m}{\Omega} = \frac{2\pi m}{\pi/8} = 16 \quad (m=1)$$

Step (2): Express the given fun. in terms of exponential

$$x(n) = 3 \cdot \frac{1}{2} \left[e^{j\pi/8 n} + e^{-j\pi/8 n} \right]$$

$$= \frac{3}{2} e^{j\pi/8 n} + \frac{3}{2} e^{-j\pi/8 n}$$

Step (3): Express the above fun. in terms of Ω

$$x(n) = \frac{3}{2} e^{j\Omega n} + \frac{3}{2} e^{-j\Omega n} \rightarrow \textcircled{1}$$

Step (4): Write the synthesis eqn

$$x(n) = \sum_{k=-N/2}^{N/2} x(k) e^{jK\Omega n}$$

Step (5): select the range of k

$$k \in -(N/2 - 1) \text{ to } N/2 \quad (\text{if } N \text{ is even})$$

$$k \in -(16/2 - 1) \text{ to } 16/2$$

$$k \in -7 \text{ to } 8$$

Step (6): Expand the synthesis eqn, by putting the values of k .

$$x(n) = \sum_{k=-7}^8 x(k) e^{jk\Omega n}$$

$$= x(-7) e^{-j7\Omega n} + \dots + x(-1) e^{-j\Omega n} + x(0) e^{j0\Omega n} + x(1) e^{j\Omega n} + x(2) e^{j2\Omega n} + \dots + x(8) e^{j8\Omega n}$$

$$x(n) = x(-7) e^{-j7\Omega n} + \dots + x(-1) e^{-j\Omega n} + x(0) + x(1) e^{j\Omega n} + \dots + x(8) e^{j8\Omega n} \rightarrow \text{①}$$

Comparing ① and ②.

$$x(-1) = 3/2$$

$$x(1) = 3/2$$

$$x(k) = \begin{cases} 3/2, & k = \pm 1 \\ 0 & -7 \leq k \leq 8 \text{ and } k \neq \pm 1 \end{cases}$$

Qn Determine the DFTS coefficients of $x(n) = \cos \pi/4 n$.

$$1) \Omega = \pi/4$$

$$N = \frac{2\pi}{\Omega} = \frac{2\pi}{\pi/4} = 8 \text{ (even)}$$

$$2) x(n) = \frac{e^{j\pi/4 n}}{2} + \frac{e^{-j\pi/4 n}}{2}$$

$$3) x(n) = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n} \rightarrow \text{①}$$

$$4. \quad x(n) = \sum_{k=-N}^{N} x(k) e^{jk\Omega n}$$

$$5. \quad k \in -\left(\frac{N}{2}-1\right) \text{ to } N/2$$

$$e^{-j\Omega n} \text{ to } 2$$

$$6. \quad x(n) = \sum_{k=-1}^2 x(k) e^{jk\Omega n}$$

$$= x(-1) e^{-j\Omega n} + x(0) + x(1) e^{j\Omega n} + x(2) e^{j2\Omega n} \rightarrow \textcircled{2}$$

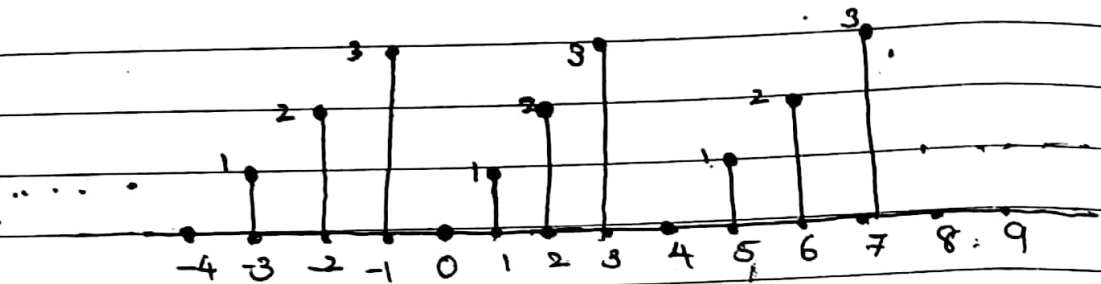
7. Comparing ① and ②

$$x(-1) = 1/2$$

$$x(1) = 1/2$$

$$\therefore x(k) = \begin{cases} 1/2 & k = \pm 1 \\ 0 & -1 \leq k \leq 2 \text{ and } k \neq \pm 1 \end{cases}$$

Qn. Determine DTFS coefficients for the periodic signal



1) Find N and Ω .

① $N = 4$

$$\Omega = \frac{2\pi}{N} = \frac{2\pi}{4} = \pi/2$$

2)

~~From the fig. $n \in \mathbb{Z}$~~

~~exists~~ From the fig. $n \in 0$ to 3 .

3) Write the analysis eqn.

$$X(k) = \frac{1}{N} \sum_{n \in \mathbb{Z}} x(n) e^{-j k \Omega n}$$

$$= \frac{1}{4} \sum_{n=0}^3 x(n) e^{-j k \Omega n}$$

$$= \frac{1}{4} \left[x(0) + x(1) e^{-j k \Omega} + x(2) e^{-j 2 k \Omega} + x(3) e^{-j 3 k \Omega} \right]$$

$$= \frac{1}{4} \left[0 + 1 e^{j k \pi / 2} + 2 e^{-j 2 k \pi / 2} + 3 e^{-j 3 k \pi / 2} \right]$$

$$x(k) = \frac{1}{4} \left[e^{-jk\pi/2} + 2e^{-jk\pi} + 3e^{-j3k\pi/2} \right]$$

$$x(0) = \frac{1}{4} [1 + 2 + 3] = \frac{6}{4} = \frac{3}{2} = 1.5$$

$$x(1) = \frac{1}{4} \left[e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2} \right]$$

$$= \frac{1}{4} \left[\cos \pi/2 - j \sin \pi/2 + 2(\cos \pi - j \sin \pi) + 3(\cos 3\pi/2 - j \sin 3\pi/2) \right]$$

$$= \frac{1}{4} [0 - j + 2(-1 - j \times 0) + 3(0 + j)]$$

$$= \frac{1}{4} [-j - 2 + 3j] = \frac{1}{4} [-2 + 2j] = \underline{\underline{-\frac{1}{2} + j\frac{1}{2}}}$$

$$= -0.5 + j0.5$$

$$x(2) = -\frac{1}{2}$$

$$x(3) = -\frac{1}{2} - j\frac{1}{2} = -0.5 - 0.5j$$

$$\therefore x(k) = \{1.5, -0.5 + j0.5, -0.5, -0.5 - j0.5\}$$

Determine the DFTS representation of the sequence $x(n) = \cos^2(\pi/4 n)$. Also sketch the magnitude spectrum.

$$x(n) = \cos^2(\pi/4 n)$$

$$= \frac{1 + \cos 2\pi/4 n}{2}$$

$$= \frac{1 + \cos \pi/2 n}{2}$$

$$1) \Omega = \pi/2$$

$$N = \frac{2\pi}{\Omega} m = \frac{2\pi}{\pi/2} m = 4 \quad (m=1)$$

$$2) x(n) = \frac{1}{2} + \frac{e^{j\pi/2 n}}{4} + \frac{e^{-j\pi/2 n}}{4}$$

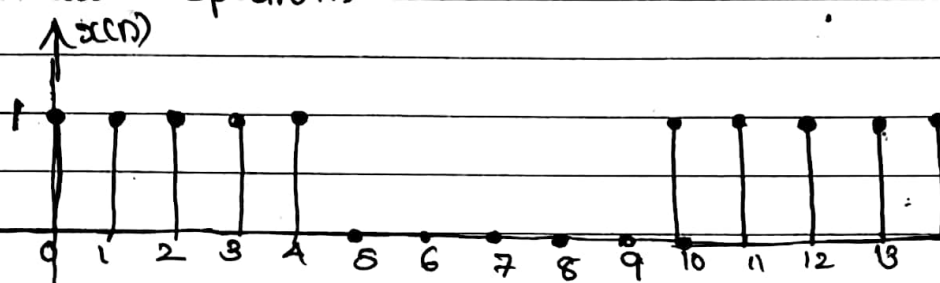
$$3) x(n) = \frac{1}{2} + \frac{e^{j\Omega n}}{4} + \frac{e^{-j\Omega n}}{4} \rightarrow \textcircled{1}$$

$$4) x(n) = \sum_{k=0}^{N-1} x(k) e^{jK\Omega n}$$

$$5) K \in \frac{-(N-1)}{2} \text{ to } \frac{N}{2}$$

$$K \in -1 \text{ to } 2$$

Determine the DTFS for $x(n)$. Also draw the magnitude spectrum.



1)

from the figure $N = 10$

$$\Omega = \frac{2\pi}{N} = \frac{2\pi}{10} = \pi/5$$

2) n ranges from 0 to 9.

$$3) X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-jk\Omega n}$$

$$= \frac{1}{10} \sum_{n=0}^9 x(n) e^{-jk\Omega n}$$

$$= \frac{1}{10} \left\{ \sum_{n=0}^4 x(n) e^{-jk\Omega n} + \sum_{n=5}^9 x(n) e^{-jk\Omega n} \right\}$$

$$= \frac{1}{10} \left\{ \sum_{n=0}^4 x(n) e^{-jk\Omega n} + 0 \right\}$$

$$5) x(n) = \sum_{k=-1}^2 x(k) e^{jk\Omega n}$$

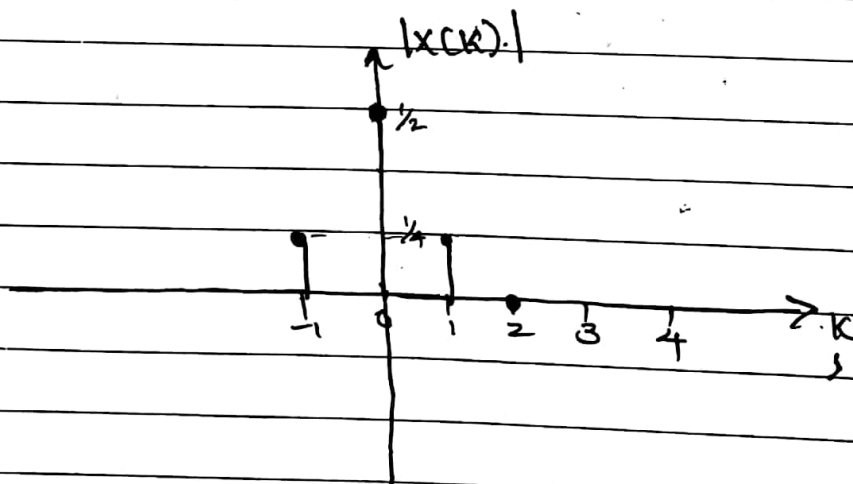
$$= x(-1) e^{-j\Omega n} + x(0) + x(1) e^{j\Omega n} + x(2) e^{2j\Omega n} \rightarrow \textcircled{2}$$

6) Comparing ① and ②

we get

$$\begin{aligned} x(-1) &= \frac{1}{4} \\ x(0) &= \frac{1}{2} \\ x(1) &= \frac{1}{4} \\ x(2) &= 0 \end{aligned}$$

magnitude spectrum:



$$= \frac{1}{10} \left[x(0) + x(1) e^{-j\Omega T} + x(2) e^{-j2\Omega T} + x(3) e^{-j3\Omega T} + x(4) e^{-j4\Omega T} \right]$$

$$X(k) = \frac{1}{10} \left[1 + x(1) e^{-j\Omega T} + x(2) e^{-j2\Omega T} + x(3) e^{-j3\Omega T} + x(4) e^{-j4\Omega T} \right]$$

$$x(0) = 0.5$$

$$x(1) = 0.1 - j0.306$$

$$x(2) = 0$$

$$x(3) = 0.1 - j0.07$$

$$x(4) = 0$$

$$x(5) = 0.1$$

$$x(6) = 0$$

$$x(7) = 0.1 - j0.07$$

$$x(8) = 0$$

$$x(9) = 0.1 - j0.307$$

Qn. Given DTFS coefficients $X(k) = \{0, 2, 0, -\frac{1}{2}, 0, 2, 0\}$

and $\Omega_0 = \frac{2\pi}{7}$. Find $x(n)$.

$$x(n) = \sum_{k=-\infty}^{\infty} X(k) e^{j k \Omega_0 n}$$

$$= \sum_{k=-3}^3 X(k) e^{j k \Omega_0 n}$$

$$= x(-3) e^{-j 3 \Omega_0 n} + x(-2) e^{-j 2 \Omega_0 n} + x(-1) e^{-j \Omega_0 n} + x(0)$$

$$+ x(1) e^{j \Omega_0 n} + x(2) e^{j 2 \Omega_0 n} + x(3) e^{j 3 \Omega_0 n}$$

$$= 2 e^{-j 2 \Omega_0 n} - \frac{1}{2} + 2 e^{j 2 \Omega_0 n}$$

$$= 2 (e^{j 2 \Omega_0 n} + e^{-j 2 \Omega_0 n}) - \frac{1}{2}$$

$$= 4 \cos(2 \Omega_0 n) - \frac{1}{2}$$

$$= 4 \cos\left(\frac{2\pi}{7} \cdot 2n\right) - \frac{1}{2}$$

$$= 4 \cos\left(\frac{4\pi n}{7}\right) - \frac{1}{2}$$

H.W

Sketch the magnitude and phase spectrum of $x(n) = 2 \sin \frac{6\pi n}{7}$.

Properties of DTFS:

1) Linearity:

$$\begin{aligned} x(n) &\xleftrightarrow{\text{DTFS}} X(k) \\ y(n) &\xleftrightarrow{\text{DTFS}} Y(k) \end{aligned}$$

$$\text{Then } z(n) = ax(n) + by(n) \xleftrightarrow{\text{DTFS}} z(k) = aX(k) + bY(k)$$

$$\text{Proof: } z(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} z(n) e^{-jk\Omega n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} [ax(n) + by(n)] e^{-jk\Omega n}$$

$$= \frac{1}{N} \left[\sum_{n=-\infty}^{\infty} a x(n) e^{-jk\Omega n} + \sum_{n=-\infty}^{\infty} b y(n) e^{-jk\Omega n} \right]$$

$$= a \underbrace{\frac{1}{N} \sum_{n=-\infty}^{\infty} x(n) e^{-jk\Omega n}}_{X(k)} + b \underbrace{\frac{1}{N} \sum_{n=-\infty}^{\infty} y(n) e^{-jk\Omega n}}_{Y(k)}$$

$$= aX(k) + bY(k)$$

2. Time shift

$$x(n) \xleftrightarrow{\text{DTFS}} X(k)$$

$$\text{Then } z(n) = x(n - n_0) \xleftrightarrow{\text{DTFS}} Z(k) = e^{-jk\Omega n_0} X(k)$$

Proof:

$$Z(k) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n) e^{-jk\Omega n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n - n_0) e^{-jk\Omega n}$$

$$\text{Put } m = n - n_0 \Rightarrow n = m + n_0$$

$$\therefore Z(k) = \frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega(m+n_0)}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega m} e^{-jk\Omega n_0}$$

$$= e^{-jk\Omega n_0} \underbrace{\frac{1}{N} \sum_{m=\langle N \rangle} x(m) e^{-jk\Omega m}}_{X(k)}$$

$$= e^{-jk\Omega n_0} X(k)$$

3. Frequency shift

$$\Downarrow x(n) \xrightarrow{\text{DTFS}} X(k)$$

$$\text{then } z(n) = e^{jK_0 n} x(n) \xrightarrow{\text{DTFS}} Z(k) = X(k - K_0)$$

$$\text{Proof: } Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} z(n) e^{-jkn}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{jK_0 n} x(n) e^{-jkn}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j(k-K_0)n}$$

$$= X(k - K_0)$$

5 Convolution:

$$\begin{aligned} x(n) &\xrightarrow{\text{DFTS}} X(k) \\ y(n) &\xrightarrow{\text{DFTS}} Y(k) \end{aligned}$$

Then

$$z(n) = x(n) * y(n) \xrightarrow{\text{DFTS}} Z(k) = N X(k) Y(k)$$

Proof

$$Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} z(n) e^{-jk\Omega n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} [x(n) * y(n)] e^{-jk\Omega n}$$

We know that

$$x(n) * y(n) = \sum_{l=0}^{n-1} x(l) y(n-l)$$

$$\therefore Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=0}^{n-1} x(l) y(n-l) \right] e^{-jk\Omega n}$$

changing the order of summations

$$Z(k) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \sum_{n=l}^{N-1} y(n-l) e^{-jk\Omega n}$$

$$\text{Put } n-l = m \Rightarrow n = m+l$$

$$\therefore Z(k) = \frac{1}{N} \sum_{l=0}^{N-1} x(l) \sum_{m=0}^{N-1-l} y(m) e^{-jk\Omega (m+l)}$$

$$\begin{aligned}
 Z(K) &= \frac{1}{N} \sum_{l=1}^{N-1} x(l) \sum_{m=1}^{N-1} y(m) e^{-jK\Omega m} e^{-jK\Omega l} \\
 &= \frac{1}{N} \sum_{l=1}^{N-1} x(l) e^{-jK\Omega l} \cdot \sum_{m=1}^{N-1} y(m) e^{-jK\Omega m} \\
 &= \underbrace{\frac{1}{N} \sum_{l=1}^{N-1} x(l) e^{-jK\Omega l}}_{X(K)} \cdot \underbrace{\sum_{m=1}^{N-1} y(m) e^{-jK\Omega m}}_{Y(K)} \\
 &= X(K) \cdot \frac{1}{N} \sum_{m=1}^{N-1} y(m) e^{-jK\Omega m} \\
 &= X(K) \cdot Y(K)
 \end{aligned}$$

6. Time reversal:

If $x(n) \xrightarrow{\text{DTFS}} X(K)$.
 Then $z(n) = x(-n) \xrightarrow{\text{DTFS}} Z(K) = X(-K)$

Proof: $Z(K) = \frac{1}{N} \sum_{n=1}^{N-1} z(n) e^{-jK\Omega n}$

$$Z(K) = \frac{1}{N} \sum_{n=1}^{N-1} x(-n) e^{-jK\Omega n}$$

Put $m = -n$

$$\therefore Z(K) = \frac{1}{N} \sum_{m=1}^{N-1} x(m) e^{+jK\Omega m}$$

$$= \frac{1}{N} \sum_{m=1}^{N-1} x(m) e^{-j(-K)\Omega m}$$

$$= X(-K)$$

7. SymmetryIf $x(n)$ is real

$$\text{then } x(-k) = x(N-k) = x^*(k).$$

$x(n)$ can be written as sum of even & odd components,
 $x(n) = x_e(n) + x_o(n)$

$$\text{If } x(n) \xrightarrow{\text{DTFS}} X(k)$$

$$\text{then } x_e(n) \xrightarrow{\text{DTFS}} \text{Re} \{X(k)\}$$

$$x_o(n) \xrightarrow{\text{DTFS}} j \text{Im} \{X(k)\}$$

ie If $x(n)$ is real and even, then its fourier coefficients are real. and while if $x(n)$ is real and odd, then its fourier coefficients are imaginary.

8. Conjugation

$$\text{If } x(n) \xrightarrow{\text{DTFS}} X(k) \\ \text{then } \cancel{x(n)} = x^*(n) \xrightarrow{\text{DTFS}} \cancel{x(k)} = X^*(-k)$$

$$\text{Proof: } Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{jk\Omega n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x^*(n) e^{-jk\Omega n} \quad X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{jk\Omega n}$$

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk\Omega n} \quad X(k) = \frac{1}{N} \left[\sum_{n=0}^{N-1} x(n) e^{jk\Omega n} \right]^*$$

$$x^*(n) = \sum_{k=0}^{N-1} X^*(k) e^{-jk\Omega n} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{j(k\Omega n)}$$

$$\text{Put } l = -k \quad \therefore x^*(n) = \sum_{l=0}^{N-1} X^*(-l) e^{j(k\Omega n)}$$

$$\therefore x^*(n) \xrightarrow{\text{DTFS}} X^*(-k)$$

9. Parseval's Theorem:

If $x(n) \xleftrightarrow{\text{DFT}} X(k)$
 then average power, $P = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |X(k)|^2$

Proof:

$$P = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) x^*(n)$$

$$\text{W.K.T } x(n) = \sum_{k=\langle N \rangle} X(k) e^{jK\Omega n}$$

$$x^*(n) = \sum_{k=\langle N \rangle} X^*(k) e^{-jK\Omega n}$$

$$\therefore P = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \cdot \sum_{k=\langle N \rangle} X^*(k) e^{-jK\Omega n}$$

changing the order of summations

$$P = \sum_{k=\langle N \rangle} X^*(k) \underbrace{\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{jK\Omega n}}_{X(k)}$$

$$P = \sum_{k=\langle N \rangle} X^*(k) X(k) = \sum_{k=\langle N \rangle} |X(k)|^2$$

where $|X(k)|^2$ is the distribution of power as a function of frequency and is defined as the power spectral density spectrum of $x(n)$.

A plot of $|X(k)|^2$ vs k is called power spectrum.

Sketch the magnitude, phase and power spectra of $x(n) = \cos\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)$

$$1) \Omega = \frac{6\pi}{13}$$

$$N = \frac{2\pi}{\Omega} = \frac{2\pi}{\frac{6\pi}{13}} = \frac{13}{3} \approx 13 \quad (m=3)$$

$$\Omega = \frac{2\pi}{N} = \frac{2\pi}{13}$$

$$2) x(n) = \frac{e^{j\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)} + e^{-j\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right)}}{2}$$

$$= \frac{e^{j\frac{6\pi}{13}n} e^{j\frac{\pi}{6}} + e^{-j\frac{6\pi}{13}n} e^{-j\frac{\pi}{6}}}{2}$$

$$3) x(n) = \frac{e^{j3\Omega n} e^{j\frac{\pi}{6}}}{2} + \frac{e^{-j3\Omega n} e^{-j\frac{\pi}{6}}}{2}$$

$$= \frac{e^{j\frac{\pi}{6}}}{2} e^{j3\Omega n} + \frac{e^{-j\frac{\pi}{6}}}{2} e^{-j3\Omega n} \rightarrow \textcircled{1}$$

$$4) x(n) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\Omega n}$$

$$5) k \in -\frac{(N-1)}{2} \text{ to } \frac{(N-1)}{2}$$

$$k \in -6 \text{ to } 6$$

$$6) x(n) = \sum_{k=-6}^6 x(k) e^{jk\Omega n}$$

②

$$x(n) = x(-6)e^{-j6\pi n} + \dots + x(-3)e^{-j3\pi n} + x(-2)e^{-j2\pi n} \\ + x(-1)e^{-j\pi n} + x(0) + x(1)e^{j\pi n} + x(2)e^{j2\pi n} + x(3)e^{j3\pi n} \\ + x(4)e^{j4\pi n} + \dots + x(6)e^{j6\pi n} \rightarrow \textcircled{2}$$

7) Comparing ① and ②

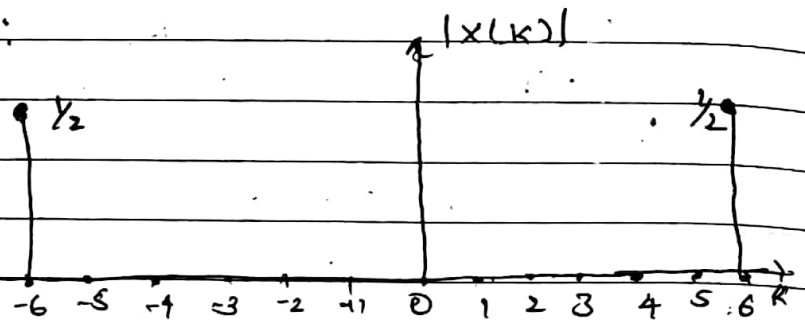
$$x(-3) = \frac{e^{-j\pi/6}}{2}$$

$$x(3) = \frac{e^{j\pi/6}}{2}$$

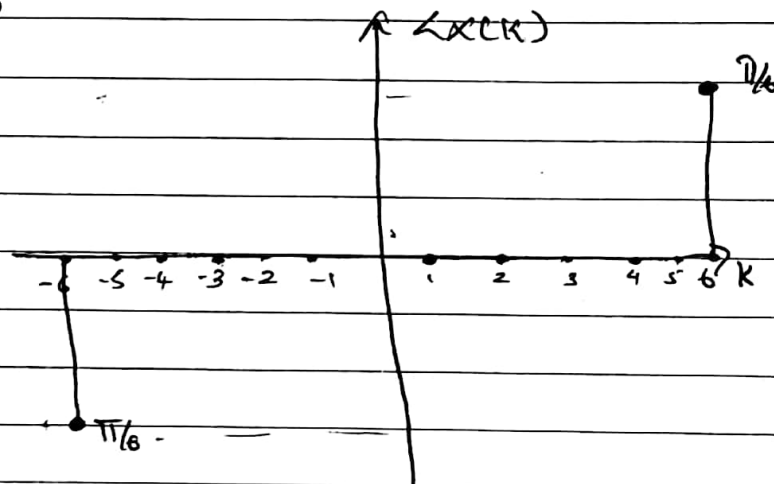
$$\therefore x(k) = \begin{cases} \frac{1}{2}e^{-j\pi/6} & k = -3 \\ \frac{1}{2}e^{j\pi/6} & k = 3 \\ 0 & \text{otherwise} \end{cases}$$

k	x(k)	x(k)	$\angle x(k)$	x(k) ²
-6	0	0	0	0
-5	0	0	0	0
-4	0	0	0	0
-3	$\frac{1}{2}e^{-j\pi/6}$	$\frac{1}{2}$	$-\pi/6$	$\frac{1}{4}$
-2	0	0	0	0
-1	0	0	0	0
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	$\frac{1}{2}e^{j\pi/6}$	$\frac{1}{2}$	$\pi/6$	$\frac{1}{4}$
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0

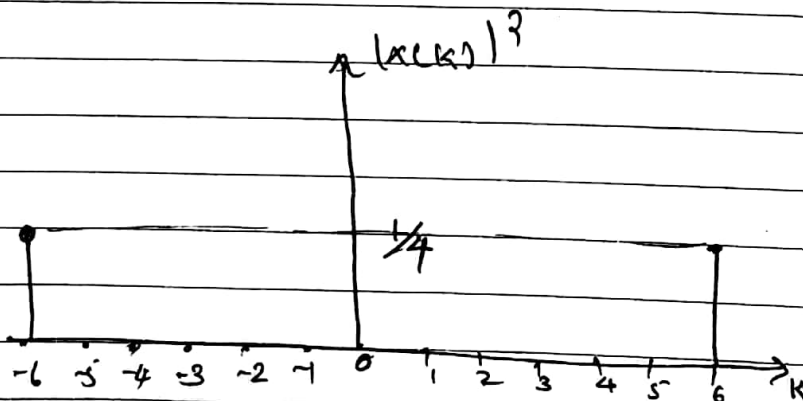
magnitude spectrum:



Phase spectrum



power spectrum



Fourier representation for aperiodic discrete time signal — Discrete time Fourier transform (DTFT).

Definition:

The DTFT of a non-periodic sequence $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow \text{Analysis eqn.}$$

The inverse DTFT of $X(\omega)$ is given by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \rightarrow \text{Synthesis eqn.}$$

Amplitude and phase spectrum:

A plot of $|X(\omega)|$ vs ω is called amplitude spectrum and a plot of $\angle X(\omega)$ vs ω is called phase spectrum.

Qn. Find the DTFT of unit impulse sequence.

$$x(n) = \delta(n)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$$

$$= e^{-j\omega n} \Big|_{n=0} = 1 //$$

Qn. Find the DTFT of $x(n) = \{1, -1, 2, 2\}$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^3 x(n) e^{-j\omega n}$$

$$= x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$= 1 - e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$$

Qn. Find the inverse DTFT of $X(\omega) = 1 + 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$
 $\rightarrow \textcircled{1}$

$$x(n) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(\omega) = \dots x(-1) e^{j\omega} + x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega} + \dots \rightarrow \textcircled{2}$$

Comparing $\textcircled{1}$ and $\textcircled{2}$

$$x(0) = 1$$

$$x(1) = 2$$

$$x(2) = 2$$

$$x(3) = 3$$

$$\therefore x(n) = \{1, 2, 2, 3\}$$

Find the DTFT of the following

1) $x(n) = [1, 2, 4, 6]$

2) $x(n) = [6, 7, 2, 1, 3]$

Qn. Find the DTFT of $x(n) = \{1, 2, 3, 2, 1\}$
 Also find $X(\Omega)$ at $\Omega = 0$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-2}^2 x(n) e^{-j\Omega n}$$

$$= x(-2) e^{+j2\Omega} + x(-1) e^{j\Omega} + x(0) + x(1) e^{-j\Omega} + x(2) e^{-j2\Omega}$$

$$= 1 \cdot e^{j2\Omega} + 2 e^{j\Omega} + 3 + 2 e^{-j\Omega} + 1 e^{-j2\Omega}$$

$$X(\Omega) \Big|_{\Omega=0} = 1 + 2 + 3 + 2 + 1 = 9 //$$

Existence of DTFT

The DTFT does not exist for all aperiodic signals. A sufficient condition for the existence of DTFT for an aperiodic sequence $x(n)$ is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

ie if a sequence $x(n)$ is absolutely summable, then DTFT exists for the sequence $x(n)$.

Qn. Find the DTFT of $x(n) = u(n)$.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=-\infty}^{\infty} |u(n)| = \sum_{n=0}^{\infty} 1 = \infty$$

It is not absolutely summable.
 $\therefore$ DTFT does not exist.

Qn. Find the DTFT of $x(n) = a^n u(n)$.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=-\infty}^{\infty} |a^n u(n)| = \sum_{n=0}^{\infty} |a|^n$$

$$= \frac{1}{1-a} < \infty \text{ for } |a| < 1$$

The sequence $x(n) = a^n u(n)$ is absolutely summable for all $|a| < 1$. $\therefore$ DTFT exists for $|a| < 1$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n u(n) e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}, \quad |a| < 1 \end{aligned}$$

Properties of DTFT:

1) Linearity:

$$\text{If } \begin{aligned} x(n) &\xleftrightarrow{\text{DTFT}} X(\omega) \\ y(n) &\xleftrightarrow{\text{DTFT}} Y(\omega) \end{aligned}$$

$$\text{Then } z(n) = ax(n) + by(n) \xleftrightarrow{\text{DTFT}} Z(\omega) = aX(\omega) + bY(\omega)$$

Proof:

$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} [ax(n) + by(n)] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a x(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} b y(n) e^{-j\omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n}$$

$$= aX(\omega) + bY(\omega).$$

2) Time Shift:

$$\text{If } x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$$

$$\text{Then } z(n) = x(n - n_0) \xleftrightarrow{\text{DTFT}} Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Proof:
$$Z(\omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\omega n}$$

Put $m = n - n_0 \Rightarrow n = m + n_0$.

$$\therefore z(n) = \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega(m+n_0)}$$

$$\left| \begin{array}{l} \sum_{m+n_0=-\infty}^{\infty} \\ m = -\infty - n_0 \\ = -\infty \end{array} \right.$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m} e^{-j\Omega n_0}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m}$$

$$= e^{-j\Omega n_0} x(\Omega)$$

3. Frequency shift

If $\xrightarrow{\text{DTFT}} x(\Omega)$
 then $z(n) = e^{j\Omega_0 n} x(n) \xrightarrow{\text{DTFT}} x(\Omega - \Omega_0)$

Proof: $z(n) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\Omega - \Omega_0)n}$$

$$= x(\Omega - \Omega_0)$$

4. Scaling:

If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$

Then $z(n) = x(an) \xleftrightarrow{\text{DTFT}} Z(\Omega) = X(\Omega/a)$.

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} x(an) e^{-j\Omega n}$$

Put $m = an \Rightarrow n = m/a$

$$Z(\Omega) = \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m/a}$$

$$\sum_{\substack{m/a = -\infty \\ m = a(-\infty) \\ = -\infty}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j(\Omega/a)m}$$

$$= X(\Omega/a)$$

5. Convolution:

If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$

$y(n) \xleftrightarrow{\text{DTFT}} Y(\Omega)$

Then $z(n) = x(n) * y(n) \xleftrightarrow{\text{DTFT}} Z(\Omega) = X(\Omega) \cdot Y(\Omega)$

Proof: $Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} [x(n) * y(n)] e^{-j\Omega n}$$

W.K.T $x(n) * y(n) = \sum_{l=-\infty}^{\infty} x(l) y(n-l)$

$$\therefore Z(\Omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x(l) y(n-l) \right] e^{-j\Omega n},$$

changing the order of summations

$$Z(\Omega) = \sum_{l=-\infty}^{\infty} x(l) \sum_{n=-\infty}^{\infty} y(n-l) e^{-j\Omega n}.$$

$$\text{Put } m = n - l \Rightarrow n = m + l.$$

$$\therefore Z(\Omega) = \sum_{l=-\infty}^{\infty} x(l) \sum_{m=-\infty}^{\infty} y(m) e^{-j\Omega(m+l)}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \sum_{m=-\infty}^{\infty} y(m) e^{-j\Omega m} \cdot e^{-j\Omega l}$$

$$= \underbrace{\sum_{l=-\infty}^{\infty} x(l) e^{-j\Omega l}}_{X(\Omega)} \cdot \underbrace{\sum_{m=-\infty}^{\infty} y(m) e^{-j\Omega m}}_{Y(\Omega)}.$$

$$\therefore Z(\Omega) = X(\Omega) \cdot Y(\Omega).$$

6. Multiplication (modulation) :

$$\begin{aligned} \text{If } x(n) &\xleftrightarrow{\text{DTFT}} X(\Omega) \\ y(n) &\xleftrightarrow{\text{DTFT}} Y(\Omega) \end{aligned}$$

$$\text{Then } z(n) = x(n) \cdot y(n) \xleftrightarrow{\text{DTFT}} Z(\Omega) = \frac{1}{2\pi} (X(\Omega) * Y(\Omega))$$

$$\text{Proof: } Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}.$$

$$= \sum_{n=-\infty}^{\infty} [x(n) \cdot y(n)] e^{-j\Omega n}.$$

W.K.T $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(n_0) e^{j\omega n_0} d\omega_0$

$$\therefore z(n) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} x(n_0) e^{j\omega n_0} d\omega_0 \right] \cdot y(n) e^{-j\omega n}$$

changing the order of summation and integration,

$$z(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(n_0) \sum_{n=-\infty}^{\infty} y(n) e^{-j(\omega - \omega_0)n} d\omega_0$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x(n_0) y(n - n_0) d\omega_0$$

$$= \frac{1}{2\pi} [x(n) * y(n)]$$

7. Conjugation

$$\text{If } x(n) \xrightarrow{\text{DFT}} X(\omega)$$

$$\text{then } z(n) = x^*(n) \xrightarrow{\text{DFT}} Z(\omega) = X^*(-\omega)$$

$$\text{Proof: } z(n) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\omega n} = \left[\sum_{n=-\infty}^{\infty} x(n) e^{j\omega n} \right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(-\omega)n} \right]^*$$

$$= \underline{\underline{X(-\omega)}}$$

8. Time Reversal:

If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$
 then $z(n) = x(-n) \xleftrightarrow{\text{DTFT}} Z(\Omega) = X(-\Omega)$

proof: $z(n) = \sum_{n=-\infty}^{\infty} z(n) e^{-j\Omega n}$
 $= \sum_{n=-\infty}^{\infty} x(-n) e^{-j\Omega n}$

put $m = -n$.

$\therefore z(n) = \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m}$

$= \sum_{m=-\infty}^{\infty} x(m) e^{-j(-\Omega)m}$

$= X(-\Omega)$

9. Symmetry:

If $x(n)$ is real, then $x(-n) = x^*(n)$

$x(n)$ can be written as sum of even and odd components $x(n) = x_e(n) + x_o(n)$.

If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$

then $x_e(n) \xleftrightarrow{\text{DTFT}} \text{Re}\{X(\Omega)\}$

$x_o(n) \xleftrightarrow{\text{DTFT}} j \text{Im}\{X(\Omega)\}$

i.e. If $x(n)$ is real and even, then its DTFT is purely real and if $x(n)$ is real and odd, then its DTFT is purely imaginary.

10 frequency differentiation (multiplication by n)

If $x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$
 then $n x(n) \xleftrightarrow{\text{DTFT}} j \frac{dX(\omega)}{d\omega}$

Proof: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

differentiating both sides w.r. to ω .

$$\frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \cdot (-jn)$$

$$\frac{1}{j} \frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} (n x(n)) e^{-j\omega n}$$

$$j \frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} n x(n) e^{-j\omega n}$$

$$\therefore n x(n) \xleftrightarrow{\text{DTFT}} j \frac{dX(\omega)}{d\omega}$$

11 Parseval's Theorem:

If $x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$
 then energy, $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

Proof:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot x^*(n)$$

W. K. T $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(\omega) e^{j\omega n} d\omega$

$$x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(\omega) e^{-j\omega n} d\omega$$

$$\therefore E = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(\omega) e^{-j\omega n} d\omega$$

changing the order of summation & integration

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(\omega) \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(\omega) x(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(\omega)|^2 d\omega$$

* In the above eqn $|x(\omega)|^2$ is known as energy density spectral density of signal $x(n)$ and E is total energy content of sequence $x(n)$.

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \delta(\omega)$$

Ans. Find the DTFT of the following signal.

$$1. \quad x(n) = e^{j20n}$$

$$x(n) = 1 \cdot e^{j20n}$$

By frequency shift property $e^{j20n} \xleftrightarrow{\text{DTFT}} 2\pi \delta(\omega - 20)$

$$2. x(n) = \cos \Omega_0 n.$$

$$= \frac{e^{j\Omega_0 n}}{2} + \frac{e^{-j\Omega_0 n}}{2}$$

$$\text{w.k.T } 1 \xleftrightarrow{\text{DTFT}} 2\pi \delta(\Omega).$$

$$1 \cdot \frac{e^{j\Omega_0 n}}{2} \xleftrightarrow{\text{DTFT}} \pi \delta(\Omega - \Omega_0)$$

$$1 \cdot \frac{e^{-j\Omega_0 n}}{2} \xleftrightarrow{\text{DTFT}} \pi \delta(\Omega + \Omega_0).$$

} By freq. shift property.

$$\therefore \cos \Omega_0 n \xleftrightarrow{\text{DTFT}} \pi \delta(\Omega - \Omega_0) + \pi \delta(\Omega + \Omega_0).$$

$$3. x(n) = n \left(\frac{1}{2}\right)^n u(n).$$

$$\text{w.k.T } a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\Omega}}$$

$$\therefore \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

By multiplication by n property.

$$n x(n) \xleftrightarrow{\text{DTFT}} j \frac{d}{d\Omega} x(\Omega).$$

$$\text{where } x(n) = \left(\frac{1}{2}\right)^n u(n), \text{ and } x(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$\therefore n x(n) = n \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{\text{DTFT}} j \frac{d}{d\Omega} \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$= j \frac{d}{d\Omega} \left[\frac{e^{j\Omega}}{e^{j\Omega} - 0.5} \right]$$

$$= \frac{j[(e^{j\Omega} - 0.5) \cdot e^{j\Omega} - e^{j\Omega} e^{j\Omega}]}{(e^{j\Omega} - 0.5)^2}$$

$$= \frac{-e^{j2\Omega} + 0.5e^{j\Omega} + e^{j2\Omega}}{(e^{j\Omega} - 0.5)^2}$$

$$= \frac{0.5e^{j\Omega}}{(e^{j\Omega} - 0.5)^2}$$

$$n a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{a e^{j\Omega}}{(e^{j\Omega} - a)^2}$$

Q $x(n) = (n+1) a^n u(n)$

$$= n a^n u(n) + a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{a e^{j\Omega}}{(e^{j\Omega} - a)^2} + \frac{1}{1 - a e^{j\Omega}}$$

$$= \frac{a e^{j\Omega}}{(e^{j\Omega} - a)^2} + \frac{e^{j\Omega}}{(e^{j\Omega} - a)}$$

$$= \frac{a e^{j\Omega} + e^{j\Omega} (e^{j\Omega} - a)}{(e^{j\Omega} - a)^2} = \frac{a e^{j\Omega} + e^{j2\Omega} - a e^{j\Omega}}{(e^{j\Omega} - a)^2}$$

$$= \frac{e^{j2\Omega}}{(e^{j\Omega} - a)^2} = \frac{1}{(1 - a e^{j\Omega})^2}$$

$$(n+1) a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{(1 - a e^{j\Omega})^2}$$

Find the DTFT of

5: $x(n) = \left(\frac{1}{2}\right)^n u(n+1)$ by using property of using DTFT eqn.

Ans:

$$x(n) = 2 \cdot \left(\frac{1}{2}\right)^{n+1} u(n+1).$$

$$a^n u(n) \xleftrightarrow{\text{DTFT}} \frac{1}{1 - a e^{-j\Omega}}$$

$$a^{n+1} u(n+1) \xleftrightarrow{\text{DTFT}} \frac{e^{-j\Omega}}{1 - a e^{-j\Omega}}$$

$$2 \cdot \left(\frac{1}{2}\right)^{n+1} u(n+1) \xleftrightarrow{\text{DTFT}} \frac{2 e^{-j\Omega}}{1 - \frac{1}{2} e^{-j\Omega}}$$

$$\begin{aligned} x(n) &\xleftrightarrow{\text{DTFT}} X(\Omega) \\ x(n-n_0) &\xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega) \end{aligned}$$

using DTFT eqn:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n+1) e^{-j\Omega n}$$

$$= \sum_{n=-1}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n}$$

$$= \left(\frac{1}{2}\right)^{-1} e^{j\Omega} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\Omega n}$$

$$= 2 e^{j\Omega} + \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\Omega}\right)^n$$

$$= 2 e^{j\Omega} + \frac{1}{1 - \frac{1}{2} e^{-j\Omega}} = \frac{2 e^{j\Omega} (1 - \frac{1}{2} e^{-j\Omega}) + 1}{(1 - \frac{1}{2} e^{-j\Omega})}$$

$$= \frac{2e^{j\Omega} - 1 + 1}{(1 - \frac{1}{2}e^{-j\Omega})}$$

$$= \frac{2e^{j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$\underline{\underline{\underline{\hspace{1cm}}}}$$

Qn. Evaluate the following function without computing $X(\Omega)$, where $X(\Omega)$ is the DTFT of the sequence $x(n) = \{1, 2, 3, 2, 2\}$. a) $X(\Omega)$ at $\Omega = 0$

b) $\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$

Ans: a) $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$$X(\Omega)|_{\Omega=0} = \sum_{n=-\infty}^{\infty} x(n) = \sum_{n=-2}^2 x(n) = 1 + 2 + 3 + 2 + 2 = 10$$

b) $\int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= 2\pi \sum_{n=-2}^2 |x(n)|^2$$

$$= 2\pi [1^2 + 2^2 + 3^2 + 2^2 + 2^2]$$

$$= 2\pi [1 + 4 + 9 + 4 + 4]$$

$$= 2\pi [22]$$

$$= 44\pi //$$

Q. Find the convolution of the signals given below using Fourier transform. $x_1(n) = (\frac{1}{2})^n u(n)$, $x_2(n) = (\frac{1}{3})^n u(n)$

Ans: ~~for~~ $y(n) = x_1(n) * x_2(n) \xrightarrow{\text{DFT}} Y(\omega) = X_1(\omega) X_2(\omega)$
(convolution property)

W.K.T $a^n u(n) \xleftrightarrow{\text{DFT}} \frac{1}{1 - a e^{-j\omega}}$

$\therefore x_1(n) = (\frac{1}{2})^n u(n) \xleftrightarrow{\text{DFT}} X_1(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$

$x_2(n) = (\frac{1}{3})^n u(n) \xleftrightarrow{\text{DFT}} X_2(\omega) = \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$

$Y(\omega) = X_1(\omega) \cdot X_2(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$

$= \frac{A}{1 - \frac{1}{2} e^{-j\omega}} + \frac{B}{1 - \frac{1}{3} e^{-j\omega}}$

$1 = A(1 - \frac{1}{3} e^{-j\omega}) + B(1 - \frac{1}{2} e^{-j\omega})$

Put $e^{-j\omega} = 3 \Rightarrow 1 = B(1 - \frac{1}{2} \cdot 3)$

$1 = B(1 - \frac{3}{2})$

$1 = B(\frac{2-3}{2})$

$1 = -\frac{1}{2} B$

$B = -2 //$

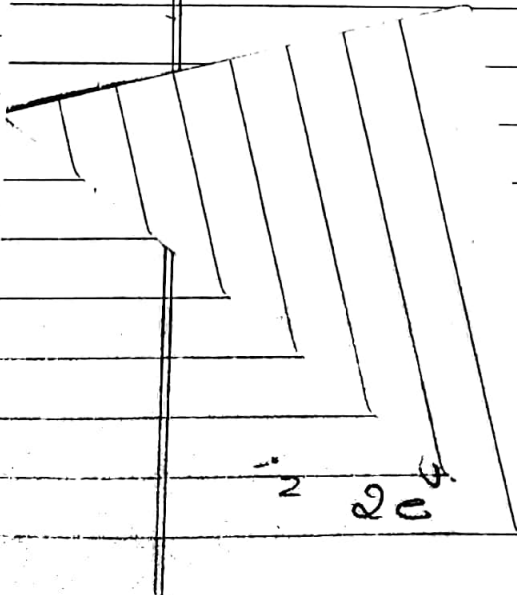
$\Rightarrow 1 = A(1 - \frac{1}{3} \cdot 2)$

$= A(1 - \frac{2}{3})$

$1 = A(\frac{3-2}{3})$

$1 = \frac{1}{3} A$

$A = 3 //$



$$\therefore Y(\omega) = 3 \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + -2 \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

Taking inverse DTFT on both sides,

$$y(n) = 3 \cdot \left(\frac{1}{2}\right)^n u(n) - 2 \cdot \left(\frac{1}{3}\right)^n u(n)$$

Qn. Verify Parseval's theorem for the sequence $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Ans: Parseval's theorem: $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

$$\text{LHS: } \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{2}\right)^n u(n) \right|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1 - \frac{1}{4}} = \frac{1}{\frac{4-1}{4}} = \frac{1}{3/4} = 4/3 = 1.33 //$$

$$\text{RHS: } \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{1 - \frac{1}{2}e^{j\omega}} \right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{1 - \frac{1}{2}e^{j\omega}} \right) \times \left(\frac{1}{1 - \frac{1}{2}e^{-j\omega}} \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left(1 - \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega} + \frac{1}{4}\right)} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 + \frac{1}{4} - \frac{1}{2}(e^{j\omega} + e^{-j\omega})} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1.25 - \cos \Omega} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1.25 - \frac{(1 - \tan^2(\Omega/2))}{1 + \tan^2(\Omega/2)}} d\Omega \quad \left[\because \cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\frac{1.25(1 + \tan^2(\Omega/2)) - (1 - \tan^2(\Omega/2))}{1 + \tan^2(\Omega/2)}} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + \tan^2(\Omega/2)}{1.25 + 1.25 \tan^2(\Omega/2) - 1 + \tan^2(\Omega/2)} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + \tan^2 \Omega/2}{0.25 + 2.25 \tan^2(\Omega/2)} d\Omega \rightarrow \textcircled{1}$$

Put $t = \tan(\Omega/2)$

diff. w. r to Ω .

$$\Omega = -\pi \Rightarrow t = \tan(-\pi/2) = -\infty$$

$$\Omega = \pi \Rightarrow t = \infty$$

$$\frac{dt}{d\Omega} = \sec^2 \Omega/2 \cdot 1/2 \Rightarrow d\Omega = \frac{2 dt}{\sec^2 \Omega/2}$$

$$= \frac{2 dt}{1 + \tan^2 \Omega/2} = \frac{2 dt}{1 + t^2}$$

$$\textcircled{1} \Rightarrow = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 + t^2)}{0.25 + 2.25 t^2} \cdot \frac{2}{(1 + t^2)} dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{2.25(t^2 + \frac{0.25}{2.25})} dt$$

$\frac{0.25}{2.25}$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{2.25(t^2 + (\frac{1}{3})^2)} dt$$

$$= \frac{1}{2.25\pi} \int_{-\infty}^{\infty} \frac{1}{t^2 + (\frac{1}{3})^2} dt$$

W. K. T

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2.25\pi} \left[3 \cdot \tan^{-1} 3t \right]_{-\infty}^{\infty}$$

$$= \frac{1}{2.25\pi} 3 [\tan^{-1}(\infty) - \tan^{-1}(-\infty)]$$

$$= \frac{1}{2.25\pi} 3 \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right]$$

$$= \frac{1}{2.25\pi} \cdot 3 \cdot \pi = \frac{3}{2.25} = 1.33 \approx \text{LHS} //$$

Frequency response in discrete time system.

The frequency response of a linear time invariant discrete time system can be obtained by applying a spectrum of input sinusoids to the system.

The frequency response gives the magnitude response and phase response of the system to the spectrum of input sinusoids.

Let the impulse response of an LTI system be $h(n)$ and the input $x(n)$ to the system be $e^{j\Omega n}$. The output of the system $y(n)$ can be obtained by using convolution sum.

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k) e^{j\Omega(n-k)} \quad \text{Or}$$

$$= \sum_{k=-\infty}^{\infty} h(k) e^{j\Omega n} e^{-j\Omega k}$$

$$= \underbrace{e^{j\Omega n}}_{\text{i/p}} \underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-j\Omega k}}_{H(\Omega)}$$

The quantity $H(\Omega)$ is the frequency response of the system.

$$x(n) = e^{j\Omega n} \quad \boxed{\text{DTI system } H(\Omega)} \quad y(n) = x(n) \cdot H(\Omega) \quad \text{Or}$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

A plot of $|H(\Omega)|$ vs Ω is called magnitude spectrum response and a plot of $\angle H(\Omega)$ vs Ω is called phase response.

Qn. Find the frequency response of the system

$$y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$$

Ans: Given $y(n) = \frac{1}{2}x(n) + x(n-1) + \frac{1}{2}x(n-2)$

Taking FT on both sides.

$$Y(\Omega) = \frac{1}{2}X(\Omega) + e^{j\Omega}X(\Omega) + \frac{1}{2}e^{j2\Omega}X(\Omega)$$

$$= X(\Omega) \left[\frac{1}{2} + e^{j\Omega} + \frac{1}{2}e^{j2\Omega} \right]$$

frequency response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{2} + e^{j\Omega} + \frac{1}{2}e^{j2\Omega}$

$$= e^{j\Omega} \left[\frac{1}{2}e^{-j\Omega} + 1 + \frac{1}{2}e^{j\Omega} \right]$$

$$= e^{j\Omega} [1 + \cos \Omega]$$

Qn. Find the frequency response of the system

$$y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = x(n) + x(n-1)$$

Ans:

Taking DFT on both sides.

$$Y(\Omega) - \frac{1}{4}e^{-j\Omega}Y(\Omega) - \frac{3}{8}e^{-j2\Omega}Y(\Omega) = X(\Omega) + e^{j\Omega}X(\Omega)$$

$$Y(\Omega) \left[1 - \frac{1}{4}e^{-j\Omega} - \frac{3}{8}e^{-j2\Omega} \right] = X(\Omega) [1 + e^{j\Omega}]$$

freq. response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 + e^{j\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{3}{8}e^{-j2\Omega}}$

Qn. Consider a causal, and stable LTI S/m whose i/p $x(n]$ and o/p $y(n]$ are related through the 2nd order difference eqn: $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n]$

- Determine the frequency response $H(\Omega)$ of the S/m
- Determine the impulse response of the S/m.

Ans: Given $y(n) - \frac{1}{6}y(n-1] - \frac{1}{6}y(n-2) = x(n)$

Taking FT on both sides.

$$Y(\Omega) - \frac{1}{6}e^{-j\Omega}Y(\Omega) - \frac{1}{6}e^{-j2\Omega}Y(\Omega) = X(\Omega)$$

$$Y(\Omega) \left[1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega} \right] = X(\Omega)$$

a) frequency response, $H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}$

b) $H(\Omega) = \frac{1}{1 - \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}} = \frac{1}{(1 - \frac{1}{2}e^{-j\Omega})(1 + \frac{1}{3}e^{-j\Omega})}$

$$= \frac{A}{(1 - \frac{1}{2}e^{-j\Omega})} + \frac{B}{(1 + \frac{1}{3}e^{-j\Omega})}$$

$$1 = A(1 + \frac{1}{3}e^{-j\Omega}) + B(1 - \frac{1}{2}e^{-j\Omega})$$

Put $e^{-j\Omega} = -3 \Rightarrow 1 = B(1 + \frac{1}{2})$

$$\frac{5}{2}B = 1$$

$$B = \frac{2}{5}$$

Put $e^{-j\Omega} = 2 \Rightarrow 1 = A(1 + \frac{2}{3})$

$$\frac{5}{3}A = 1$$

$$A = \frac{3}{5}$$

$$\therefore H(\Omega) = \frac{3}{5} \frac{1}{(1 - \frac{1}{2}e^{-j\Omega})} + \frac{2}{5} \frac{1}{(1 + \frac{1}{3}e^{-j\Omega})}$$

Taking inverse DFT

impulse response, $h(n) = \frac{3}{5} \left(\frac{1}{2}\right)^n u(n) + \frac{2}{5} \left(-\frac{1}{3}\right)^n u(n)$

$$\left(\because a^n u(n) \xleftrightarrow{\text{DFT}} \frac{1}{1 - ae^{-j\Omega}} \right)$$

Qn. Consider a discrete-time LTI s/m with impulse response $h(n) = (\frac{1}{2})^n u(n)$. Use Fourier transform determine the response to the following signals.

a) $x(n) = (\frac{3}{4})^n u(n)$

b) $x(n) = (-1)^n u(n)$

Ans:

Given $h(n) = (\frac{1}{2})^n u(n)$

$$H(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

a) $x(n) = (\frac{3}{4})^n u(n)$

$$X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega) \cdot X(\Omega)$$

$$= \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \cdot \frac{1}{1 - \frac{3}{4}e^{-j\Omega}} = \frac{A}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{3}{4}e^{-j\Omega}}$$

$$A = 3, B = -2$$

$$\therefore Y(\Omega) = 3 \cdot \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} - 2 \cdot \frac{1}{1 - \frac{3}{4}e^{-j\Omega}}$$

Taking inverse DTFT

$$y(n) = 3 \left(\frac{1}{2}\right)^n u(n) - 2 \left(\frac{3}{4}\right)^n u(n) //$$

b) $x(n) = (-1)^n u(n)$

$$X(\Omega) = \frac{1}{1 + e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega) \cdot X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \cdot \frac{1}{1 + e^{-j\Omega}} = \frac{A}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 + e^{-j\Omega}}$$

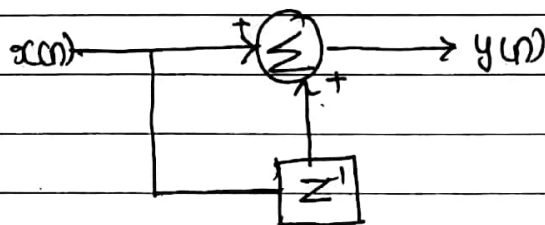
$$A = \frac{2}{5}, B = \frac{1}{5}$$

$$\therefore Y(\Omega) = \frac{2}{5} \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{1}{5} \frac{1}{1 + e^{-j\Omega}}$$

Taking inverse DTFT

$$y(n) = \frac{2}{5} \left(\frac{1}{2}\right)^n u(n) + \frac{1}{5} (-1)^n u(n) //$$

Qn. Consider the discrete time LTI S/m shown in fig. given below.



- Find the frequency response of the S/m.
- Find the impulse response of the S/m.
- Find the 3dB Bandwidth of the S/m.
- Sketch the magnitude and phase response.

Ans: from the fig. $y[n] = x[n] + x[n-1]$
 $Y(\omega) = X(\omega) + e^{-j\omega} X(\omega)$
 $= X(\omega) [1 + e^{-j\omega}]$

a) frequency response, $H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + e^{-j\omega}$
 $= 1 + \cos \omega - j \sin \omega$

b) Impulse response means when $x[n] = \delta[n]$.
 $y[n] = h[n]$

$\therefore h[n] = \delta[n] + \delta[n-1]$

c) 3 dB Bandwidth

$$|H(\omega)_{3dB}| = \frac{1}{\sqrt{2}} |H(\omega)|$$

$$= \frac{1}{\sqrt{2}} 2 = \sqrt{2}$$

$$\sqrt{(1 + \cos \omega_{3dB})^2 + \sin^2 \omega_{3dB}} = \sqrt{2}$$

$$\sqrt{1 + 2\cos\Omega + \underbrace{\cos^2\Omega + \sin^2\Omega}_{=1}} = \sqrt{2}$$

$$\cos\pi = -1$$

$$\cos\pi/2 = 0$$

$$\cos 0 = 1$$

$$\cos\pi/4 = 1/\sqrt{2}$$

$$\sqrt{2 + 2\cos\Omega_{3dB}} = \sqrt{2}$$

$$2 + 2\cos\Omega_{3dB} = 2$$

$$2\cos\Omega_{3dB} = 0$$

$$\cos\Omega_{3dB} = 0 \Rightarrow \Omega_{3dB} = \cos^{-1}(0) = \pi/2 //$$

d) Ω $|H(\omega)| = \sqrt{2 + 2\cos\Omega}$ $\angle H(\omega) = \tan^{-1}\left(\frac{-\sin\Omega}{1 + \cos\Omega}\right)$
 $= -\tan^{-1}\left(\frac{\sin\Omega}{1 + \cos\Omega}\right)$

$$-\pi$$

$$0$$

$$\tan^{-1}(0) = \tan^{-1}(\infty) = 90$$

$$-\pi/4$$

$$1.84$$

$$+22.5$$

$$-\pi/2$$

$$1.414$$

$$+45$$

$$0$$

$$2$$

$$0$$

$$\pi/2$$

$$1.414$$

$$-45$$

$$\pi/4$$

$$1.84$$

$$-22.5$$

$$\pi$$

$$0$$

$$-90$$